PSEUDOSPECTRA OF SEMICLASSICAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. We consider operators $-\Delta + X$ where X is a constant vector field, in a bounded domain and show spectral instability when the domain is expanded by scaling. More generally, we consider semiclassical elliptic boundary value problems which exhibit spectral instability for small values of the semiclassical parameter h, which should be thought of as the reciprocal of the Peclet constant. This instability is due to the presence of the boundary: just as in the case of $-\Delta + X$, some of our operators are normal when considered on $\mathbb{R}^d$. We characterize the semiclassical pseudospectrum of such problems as well as the areas of concentration of quasimodes. As an application we prove a result about exit times for diffusion processes in bounded domains. We also demonstrate instability for a class of spectrally stable nonlinear evolution problems that are associated to these elliptic operators.

1. INTRODUCTION

For many non-normal operators, the size of the resolvent is not connected to the distance to the spectrum and is a measure of the spectral instability. The level sets of the norm of the resolvent are referred to as the *pseudospectrum*. The study of the pseudospectrum has been a topic of interest both in applied mathematics (see [2],[22], [10], and numerous references given there) and the theory of partial differential equations (see, for example [23],[3],[18],[9], [11], [12]).

The problem of characterizing pseudospectra for semiclassical partial differential operators acting on Sobolev spaces on $\mathbb{R}^d$ started with [2]. Dencker, Sjöstrand, and Zworski gave a more complete characterization for these pseudospectra in [4] by proving that, for operators with Weyl symbol p, if $(p - z)(x_0, \xi_0) = 0$ and $i\{p, \bar{p}\}(x_0, \xi_0) < 0$ then z is in the pseudospectrum of p^w and, moreover, there exists a quasimode at z in the following sense. There exists $u \in H_h^2$ with $\|u\|_{L^2} = 1$ and $WF_h(u) = \{(x_0, \xi_0)\}$ such that

$$(p^w - z)u = O_{L^2}(h^\infty).$$

In [18], Pravda-Starov extended the results of [4] and gave a slightly different notion of pseudospectrum.

In this paper, we examine the size of the resolvent for operators defined on bounded domains $\Omega \subset \mathbb{R}^d$ with C^∞ boundary. Let

$$(1.1) \qquad\qquad P = (hD)^2 + i\langle X, hD \rangle \quad D_j := \frac{1}{i}\partial_j.$$

Here, h can be thought of as the inverse of the Peclet constant. We are interested in determining the pseudospectrum of the Dirichlet operator P on Ω in the following sense. We wish to find

$z \in \mathbb{C}$ and $u \in H_h^2$ such that

(1.2)
$$\begin{cases} (P-z)u = O_{L^2}(h^\infty) & x \in \Omega, \\ u|_{\partial\Omega} = 0\,, \|u\|_{L^2} = 1. \end{cases}$$

The collection of such z will be denoted $\Lambda(P,\Omega)$ and the pseudospectrum of (P,Ω) is $\overline{\Lambda(P,\Omega)}$. A solution to (1.2) will be called a quasimode for z. We restrict our attention to the case where X is constant so that there are no quasimodes given by the results of [4].

We characterize $\overline{\Lambda(P,\Omega)}$ for such boundary value problems. In addition, we characterize the semiclassical essential support of quasimodes where the essential support is defined as

Definition 1. *The essential support of a family of h-dependent functions $u = u(h)$ is given by*

$$ES_h(u) := \bigcap_{U \in \mathcal{A}} U\,, \quad \mathcal{A} := \{U \subset \overline{\Omega} : \text{ if } \chi \in C^\infty(\overline{\Omega}),\ \chi \equiv 1 \text{ on } U,\ \text{then } (1-\chi)u = O_{L^2}(h^\infty)\}.$$

In order to accomplish the characterization, we need the following analogue of convexity similar to that used for planar domains in [16]

Definition 2. *A set $A \subset B$ is relatively convex in B if, defining,*

$$L_{x,y} = \{tx + (1-t)y : t \in [0,1]\},$$

we have that for all $x, y \in A$, $L_{x,y} \subset B$ implies $L_{x,y} \subset A$.

We also need an analogue of the convex hull in this setting

Definition 3. *For $A \subset B$, we define the convex hull of A relative to B by*

$$Coh_B(A) = \bigcap_{C \in \mathcal{A}} C\,, \quad \mathcal{A} := \{C : A \subset C \text{ and } C \text{ is relatively convex in } B\}.$$

(See Figure 3 for an example.)

If $A \nsubseteq B$, then define

$$Coh_B(A) = Coh_B(A \cap B).$$

Remark: In the case that B is convex, these definitions coincide with the usual notions of convexity.

We define subsets of $\partial\Omega$ similar to those in [15],
(1.3)
$$\partial\Omega_- = \{x \in \partial\Omega : \langle X, \nu \rangle < 0\}, \quad \partial\Omega_+ = \{x \in \partial\Omega : \langle X, \nu \rangle > 0\}, \quad \partial\Omega_0 = \{x \in \partial\Omega : \langle X, \nu \rangle = 0\}$$
$$\Gamma_+ = \partial\Omega_+ \cup \partial\Omega_0,$$

and show

Theorem 1. *Let P as in (1.1), and $\Omega \subset \mathbb{R}^d$ a domain with C^∞ boundary. Then,*

(1) $\overline{\Lambda(P,\Omega)} = \{z \in \mathbb{C} : \operatorname{Re} z \geq (\operatorname{Im} z)^2 |X|^{-2}\}$.
(2) *For all quasimodes u,*

$$ES_h(u) \subset \overline{Coh_{\overline{\Omega}}(\Gamma_+)} \cap \partial\Omega \quad and \quad ES_h(u) \cap \overline{\partial\Omega_+} \neq \emptyset.$$

(3) *For each point $x_0 \in \partial\Omega_+$, there exists*

$$W_{x_0} \subset \{z \in \mathbb{C} : \operatorname{Re} z > (\operatorname{Im} z)^2 |X|^{-2}\}$$

such that W_{x_0} is open and dense in $\overline{\Lambda(P, \Omega)}$ and for each $z \in W_{x_0}$, there is a quasimode u for z with $ES_h(u) = x_0$. Morover, if $\partial\Omega$ is real analytic near x_0, then these quasimodes can be constructed with $Pu = O(e^{-c/h})$.

(4) *Let $x_0 \in \partial\Omega_-$. Suppose that $\partial\Omega$ is strictly convex or strictly concave at x_0. Then, for any quasimode u, $x_0 \notin ES_h(u)$.*

(5) *If $\Omega \subset \mathbb{R}^2$, and u is a quasimode, then $ES_h(u) \subset \Gamma_+$.*

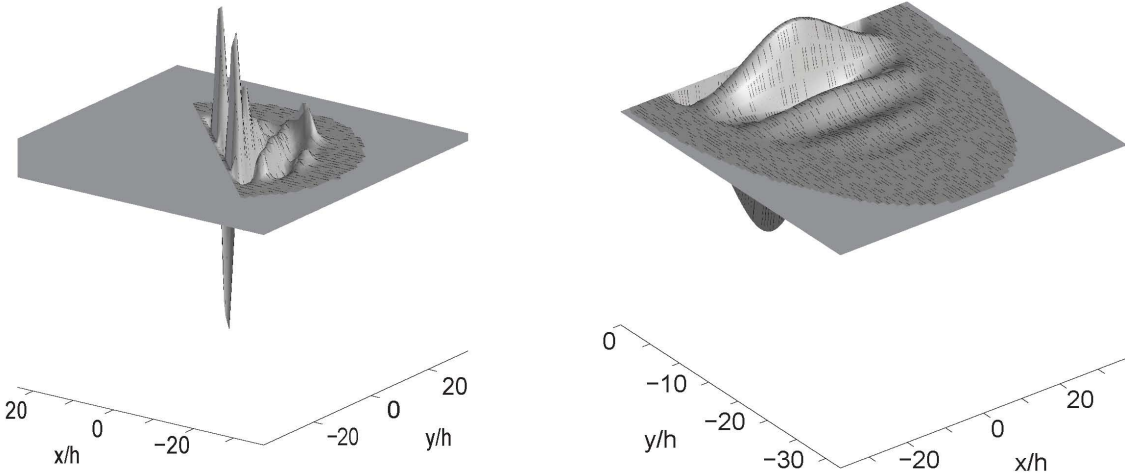**Remark:** If Ω is convex then Theorem 1 gives that $ES_h(u) \subset \Gamma_+$.

FIGURE 1. The figure shows an example of the imaginary part of a quasimode constructed in Propostion 2. On the left, the boundary forms an angle of $\pi/4$ with $X = \partial_{x_1}$, and on the right the boundary is normal to ∂_{x_1}. In both cases, $z = 1+i/2$.

Note: We will refer to $\partial\Omega_+$, $\partial\Omega_0$, and $\partial\Omega_-$ as the illuminated, glancing, and shadow sides of the boundary, respectively. Figure 3 shows examples of these subsets in a 2 dimensional domain.

It is easily seen that the spectrum of (P, Ω) is contained in $\{z \in \mathbb{C} : \operatorname{Re} z \geq c > 0, \ \operatorname{Im} z = 0\}$. Thus, Theorem 1 shows that the pseuodspectrum of (P, Ω) is far from its spectrum and hence that the size of the resolvent is unstable in the semiclassical limit. (Figure 2 shows the spectrum and pseudospectrum of (P, Ω) in an example.)

For a large class of nonlinear evolution equations this type of behavior has been proposed as an explanation of instability for spectrally stable problems. Celebrated examples include the plane Couette flow, plane Poiseuille flow and plane flow – see Trefethen-Embree [22, Chapter 20] for discussion and references. Motivated by this, we consider the mathematical question of evolution involving a small parameter h (in fluid dynamics problem we can think of h as the reciprocal of
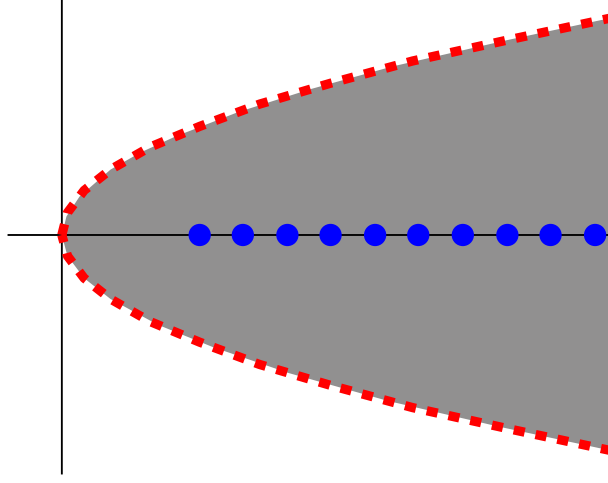
FIGURE 2.   The figure shows the pseudospectrum and spectrum of (P, Ω) for $|X| = 1$. The pseudospectrum is the shaded region, the spectrum is shown as blue circles, and the curve $\operatorname{Re} z = (\operatorname{Im} z)^2$ is shown in dashed red.
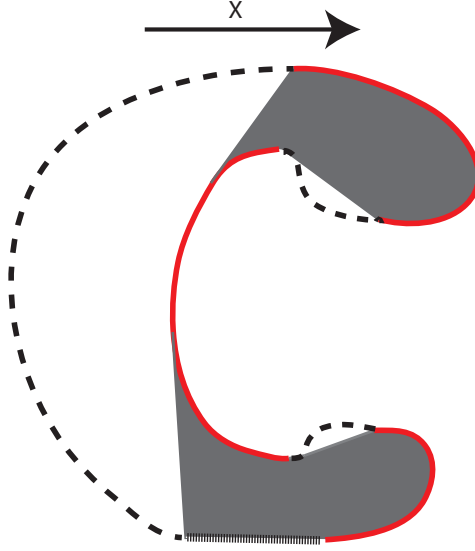

FIGURE 3.   The figure shows an example domain $\Omega \subset \mathbb{R}^2$. $\partial \Omega_+$ is shown in the solid red line, $\partial \Omega_0$ in the dotted black line, and $\partial \Omega_-$ in the dashed black line. The region $\overline{Coh_{\overline{\Omega}}(\Gamma_+)}$ is shaded in.

the Reynolds number) in which the linearized operator has spectrum lying in $\operatorname{Re} z < -\gamma_0 < 0$, uniformly in h, yet the the solutions of the nonlinear equation blow up in short time for data of size $O(\exp(-c/h))$.

We examine the behavior of the following nonlinear evolution problem

$$(1.4) \qquad \begin{cases} (h\partial_t + (P - \mu))u - u^3 = 0, & t \geq 0, \quad x \in \Omega \subset \mathbb{R}^d, \\ u|_{\partial\Omega} = 0 & u(x,0) = u_0(x), \end{cases}$$

and interpret it in terms of the pseudospectral region of $P - \mu$.

We have the following analog of what is shown in [8] and [20]

Theorem 2. *Fix $\mu > 0$. Then, for*

$$0 < h < h_0 \,,$$

where h_0 is small enough, and each $\delta > 0$, there exists

$$u_0 \in C_c^\infty(\mathbb{R}^n)\,, \quad u_0 \geq 0\,, \quad \|u_0\|_{C^p} \leq \exp\left(-\frac{1}{Ch}\right)\,, \quad p = 0, 1, \dots,$$

such that the solution to (1.4) with $u(x,0) = u_0(x)$, satisfies

$$\|u(x,t)\|_{L^\infty} \longrightarrow \infty, \quad t \longrightarrow T\,,$$

where $T \leq \delta$.

As an application of Theorem 1, we consider diffusion processes on bounded domains. Specifically, we examine hitting times

$$\tau_X = \inf\{t \geq 0 : X_t \in \partial\Omega\}$$

for processes of the form

$$dX_t = b(X_t) + \sqrt{2h}dB_t \quad X_0 = x_0(h)$$

where B_t is standard Brownian motion in d dimensions and $x_0(h) \to x_0 \in \partial\Omega_+$. We show that, for $\partial\Omega_+$ analytic near x_0, the log moment generating function of τ_x does not decay as $h \to 0$ for $|x_0(h) - x_0| = Ch^N$, and moreover, for such $x_0(h)$, there exists $\delta > 0$ such that for all $a > 1$, there is a sequence $s(h) > \delta/2$ with

$$c_a \min(e^{-a(s(h)-\delta)/h}, 1) \leq P\left(\tau_X \geq \frac{\delta}{2\lambda}\right).$$

2. Outline of the Proof

In this section, we explain the ideas of the proof of Theorem 1. We also describe the structure of the paper.

Our starting point is to prove that if $|p(x, \xi)| > C\langle\xi\rangle^2$ then (1.2) has an inverse that is bounded independent of h on semiclassical Sobolev spaces. We do this via a construction of Calderón projectors for such boundary value problems, adapted from [14, Chapter 20]. From this fact, it follows easily that

$$\Lambda(P, \Omega) \subset \{\mathrm{Re}\, z \geq (\mathrm{Im}\, z)^2 |X|^{-2}\}.$$

Next, we show that

$$\overline{\Lambda(P, \Omega)} = \{\mathrm{Re}\, z \geq (\mathrm{Im}\, z)^2 |X|^{-2}\}.$$

In particular, we construct quasimodes near points $x_0 \in \partial\Omega_+$. To do this, we use a WKB method adapted to Dirichlet boundary value problems. Motivated by the fact that, in one dimension, eigenfunctions of the Dirichlet realization of P are of the form $e^{ax/h}sin(x)$, we look for solutions of the form

$$e^{i\varphi_1/h}a - be^{i\varphi_2/h}$$

and derive formulae for WKB expansions of a and b. In order to complete this construction, we have to solve a complex eikonal equation for φ. This is accomplished by reducing the problem to solving for the Taylor series expansions of a, b, and φ_i near the point x_0. Figure 1 shows examples of quasimodes constructed using this method.

Our last task is to restrict the essential support of quasimodes u. The main idea is to prove a Carleman type estimate for solutions to (1.2). This estimate will give us control of solutions outside relatively convex sets containing Γ_+. Hence any quasimode is essentially supported inside such a convex set. The last ingredient in the proof is a result adapted from [5] on propagation of semiclassical wavefront sets for solutions of (1.2). We show that the wavefront set of a quasimode is invariant under the leaves generated by $H_{\mathrm{Im}\, p}$ and $H_{\mathrm{Re}\, p}$. Then, we show that there exist convex sets containing $\partial\Omega_0$ and $\partial\Omega_+$ which do not extremize $\langle X, x\rangle$ inside Ω. Hence, since $H_{\mathrm{Im}\, p} = \langle X, \partial\rangle$, we are able to put this together with the propagation results to show that

$$ES_h(u) \subset \overline{Coh_{\overline{\Omega}}(\Gamma_+)} \cap \partial\Omega.$$

The paper is organized as follows. In section 3 we prove results on Calderón projectors adapted from [14, Chapter 20]. Section 4 contains the construction of quasimodes via a boundary WKB method. In section 5, we adapt results of Duistermaat and Hörmander in [5, Chapter 7] on propagation of wavefront sets to the semiclassical setting. In section 6, we prove a Carleman type estimate that will be used in section 7 to derive restrictions on the essential support of quasimodes. Section 8 contains the proof of Theorem 2. Finally, section 9 applies some of the results of Theorem 1 to exit times for diffusion processes.

3. Calderón Projectors for Elliptic Symbols

We follow Hörmander's construction in the classical setting [14, Chapter 20] to construct a Calderón projector for strongly elliptic semiclassical boundary value problems. We seek to find

an inverse for the following elliptic boundary value problem

$$(3.1) \qquad \begin{cases} Pu = f & \text{in } \Omega, \\ B_j u = g_j, \ j = 1, ..., m-1 & \text{on } \partial\Omega, \end{cases}$$

where $P = p(x, hD)$ has p elliptic (in the semiclassical sense) and is a differential operator of order m, and B_j are differential operators on the boundary. We can assume without loss of generality that the order of B_j transversal to $\partial\Omega$ is less than m since the ellipticity of P implies that $Pu = f$ can be solved for $(hD_\nu)^m$. We refer the reader to [14, Definition 20.1.1] for the definition of classical boundary problem ellipticity. (Note that in our applications, B_j is the identity.) In particular, we prove the following Proposition

Proposition 1. *Let $p(x, \xi)$ have $|p(x, \xi)| \geq c\langle\xi\rangle^m$ and (3.1) be classically elliptic. Then, for h small enough, the system (3.1) has a bounded inverse*

$$P_\Omega^{-1} : H_h^{s-m}(\Omega) \oplus_{j=0}^{m-1} H_h^{s-j-\frac{1}{2}}(\partial\Omega) \to H_h^s(\Omega).$$

3.1. **Pseudospectra Lie Inside the Numerical Range.** Observe that Proposition 1 gives that pseudospectra for elliptic boundary value problems must lie inside the numerical range of $p(x, \xi)$. In the special case of P as in (1.1), we have that $P = p^w$ where $p(x, \xi) = |\xi|^2 + i\langle X, \xi\rangle$. By Proposition 1, we have that, if $P - z$ is strongly elliptic, i.e. $|p(x, \xi) - z| \geq c\langle\xi\rangle^2$, then no quasimodes for z exist.

Using this, observe that $p - z = 0$ implies the following

$$\langle X, \xi\rangle = \text{Im } z, \quad |\xi|^2 - \text{Re } z = 0.$$

Hence,

$$(3.2) \qquad \begin{cases} \xi = \text{Im } z X^\flat |X|^{-1} + w^\flat & \text{where } \langle X, w\rangle = 0, \\ |w|^2 = \text{Re } z - |X|^{-2}(\text{Im } z)^2. \end{cases}$$

This implies that

$$(3.3) \qquad \Lambda(P, \Omega) \subset \{z \in \mathbb{C} : \text{Re } z \geq |X|^{-2}(\text{Im } z)^2\}.$$

3.2. **Proof of Proposition 1.** We will follow Hörmander's proof from [14, Chapter 20] almost exactly. We present the proof in detail to provide a reference for Calderón projectors in the semiclassical setting. Note that in the semiclassically elliptic setting, the construction yields a true inverse for the boundary value problem.

First, extend P to a neighborhood, $\hat{\Omega}$ of $\overline{\Omega}$ so that P is strongly elliptic on $\hat{\Omega}$. Then, define $T = P^{-1}$ where T exists and is a pseudodifferential operator since p is semiclassiically elliptic i.e. T is given by

$$T = \left[\frac{1}{p}(x, hD)p(x, hD)\right]^{-1} \frac{1}{p}(x, hD) \in \Psi^{-m}.$$

(The fact that T is pseudodifferential follows from Beals' Theorem, (see for example [24, Section 9.3.4 and remark after Theorem 8.3])) We will construct the Calderón projector locally and hence reduce to the case where $\partial\Omega = \{x_1 = 0\}$ by a change to semigeodesic coordinates for $\partial\Omega$ and an application of a partition of unity.

For $u \in C^\infty(\overline{\Omega})$, define

$$\gamma u := (u, (hD_1)^1 u, ..., (hD_1)^{m-1})|_{\partial\Omega} \in C^\infty(\partial\Omega).$$

In $\partial\Omega \times [0,1)$, we have

$$P = \sum_0^m P_j (hD_1)^j$$

where P_j are semiclassical differential operators of order $m-j$ in $\partial\Omega$ depending on the parameter x_1. We denote the principal symbol of P_j by $\sigma(P_j) = p_j$. Next, let u^0 denote extension by 0 off of Ω^o. We have

$$Pu^0 = (Pu)^0 + P^c\gamma u$$

where for $U = (U_0, ..., U_{m-1}) \in C^\infty(\partial\Omega)$, and

$$P^c U = i^{-1} \sum_{j<m} hP_{j+1} \sum_{k \leq j} U_{j-k} \otimes (hD_1)^k \delta,$$

where δ is the Dirac mass at $x_1 = 0$. Then,

(3.4) $$u^0 = T[(Pu)^0 + P^c\gamma u].$$

Next, define Q, the Calderón projector, for $U \in C^\infty(\partial\Omega, \mathbb{C}^m)$, by $QU := \gamma T P^c U$. Then, for $k = 0, ..., m-1$,

$$(QU)_k = \sum_0^{m-1} Q_{kl} U_l$$

with

$$Q_{kl} U_l = \sum_{j=0}^{m-1-l} hi^{-1}(hD_1)^k T P_{j+l+1} U_l \otimes (hD_1)^j \delta|_{\partial\Omega}.$$

(Note that the boundary values are taken from Ω^o.) Therefore, we have Q_{kl} are pseudodifferential operators in $\partial\Omega$ of order $k-l$ with principal symbol

(3.5) $$q_{kl}(x', \xi') = (2\pi i)^{-1} \int^+ \sum_{j=0}^{m-l-1} \xi_n^{k+j} p(x', 0, \xi)^{-1} p_{j+l+1}(0, x', \xi') d\xi_1,$$

where the $\int^+$ denotes the sum of residues for $\operatorname{Im} \xi_n > 0$.

Lemma 1. *Q is a projection in the space of Cauchy data in the sense that $Q^2 - Q = 0$. If we identify solutions of the ordinary differential equation $p(0, x', hD_1, \xi')v = 0$ with the Cauchy data $(v(0), ..., (hD_1)^{m-1}v(0))$ then, $q(x', \xi')$ is for $\xi' \neq 0$ identified with the projection on the subspace M^+ of solutions exponentially decreasing on $\mathbb{R}_+$ along the subspace M^- of solutions exponentially decreasing in $\mathbb{R}_-$.*

Proof. Let $u = TP^cU$, by (3.4),

$$Pu = PTP^c U = P^c U = 0 \text{ in } \Omega^o.$$

Also, by (3.4),

$$QU = \gamma(TP^c U)^o = \gamma T((PTP^c U)^o + P^c \gamma u) = \gamma T(P^c U)^0 + \gamma T P^c \gamma T P^c U = Q^2 U$$

since $P^c U = 0$ in Ω^o. Hence, $Q^2 - Q = 0$ and Q is a projection.

To see the second part of the claim, let U as above. Then, the inverse semiclassical Fourier transform

$$v(x_1) = (2\pi hi)^{-1} h \int p(0, x', \xi)^{-1} \sum_{j+l<m} p_{j+l+1}(0, x', \xi') \xi_1^j U_l e^{ix_1\xi_1/h} d\xi_1$$

is in $\mathcal{S}'$ and satisfies

$$p(0, x', \xi', hD_1)v = hi^{-1} \sum_{j+l<m} p_{j+l+1}(x', 0, \xi') U_l (hD_1)^j \delta$$

and hence v concides for $x_1 > 0$ with an element $v^+ \in M^+$ and for $x_1 < 0$ with an element $v^- \in M^-$. For $x_1 = 0$, we have the jump condidition

$$(hD_1)^k(v^+ - v^-) = U_k.$$

Then, (3.5) gives

$$q(x', \xi')U = (v^+(0), ..., (hD_1)^{m-1}v^+(0))$$

and $qU = U$ implies $v^- = 0$, i.e. U Is the Cauchy data of a solution in M^+. Also, if $qU = 0$, U is the Cauchy data for $v \in M^-$ and we have proven the claim. $\qquad\square$

Now that we have Q defined locally, we can extend it to a global pseudodifferential operator on $\partial\Omega$ by taking a locally finite partition of unity, χ_j subordinate to U_j the semigeodesic coordinate patches and letting

$$Q = \sum_j \chi_j Q_j.$$

To complete the proof, we need the following lemma

Lemma 2. *Let Y be a compact manifold without boundary. Suppose that $Q \in \Psi^0(Y, \mathbb{C}^k, \mathbb{C}^k)$ with $Q^2 - Q = 0$, and $B \in \Psi^m(Y; \mathbb{C}^k, \mathbb{C}^n)$ with symbols q and b respectively. Then,*

(1) *if $b(y, \eta)$ restricted to $q(y, \eta)\mathbb{C}^k$ is surjective for all $(y, \eta) \in T^*(Y)$, then one can find $S \in \Psi^{-m}(Y; \mathbb{C}^n, \mathbb{C}^k)$ such that*

$$BS = I_n + O_{\Psi^0}(h), \quad QS = S$$

(2) *if $b(y, \eta)$ restricted to $q(y, \eta)\mathbb{C}^k$ is injective for all $(y, \eta) \in T^*(Y)$, then one can find $S' \in \Psi^{-m}(Y; \mathbb{C}^n, \mathbb{C}^k)$ and $S'' \in \Psi^0(Y; \mathbb{C}^k, \mathbb{C}^k)$ such that*

$$S'B + S'' = I_k + O(h), \quad S''Q = 0$$

(3) *if $b(y, \eta)$ restricted to $q(y, \eta)\mathbb{C}^k$ is bijective, then S, S', S'' are uniquely determined mod $O(h)$ and $S' = S + O(h)$.*

Proof. To prove the first claim, observe the bq is surjective. Hence there exists a right inverse c. Thus, $bqc = I$ and hence $BQC = I + O(h)$ and $Q(QC) = QC$. Hence, the first claim follows from letting $S = QC$.

To prove the second claim, observe that $b \oplus (I - q)$ is injective and hence has a left inverse (t', t''). Thus, letting $S' = T'$ and $T'' = S''$, we have the claim.

To prove the third claim, just observe that

$$S' = S'BS + O(h) = S'BS + S''(S - QS) + O(h) = S'BS + S''S + O(h) = S + O(h).$$

$\square$

Q and B satisfy the hypotheses of the part (iii) of the previous lemma by [14, Theorem 19.5.3]. Therefore, using this on Q and B from above, we obtain S and S'' as described.

As in [14, Chapter 20], we have that $R : (f,g) \mapsto (I + TP^cS''\gamma)Tf^0 + TP^cSg$ is a candidate for an approximate left and right inverse modulo $O(h)$ errors.

Therefore, in order to show that R is both an approximate left and right inverse for (3.1), all we need is the following lemma wich follows from a rescaling of [14, Proposition 20.1.6] along with the fact that $PT = I$ with no remainder. We include the proof here for convenience.

Lemma 3. *If $s \geq m$ and $f \in C^\infty$, then*

$$(3.6) \qquad \qquad \|(Tf)^o\|_{H_h^s} \leq C\|f\|_{H_h^{s-m}}.$$

If $U = (U_0, ..., U_{m-1}) \in C^\infty(\partial\Omega)$, we have for any s

$$(3.7) \qquad \qquad \|(TP^cU)^o\|_{H_h^s} \leq C \sum_0^{m-1} \|U_j\|_{H_h^{s-j-\frac{1}{2}}}.$$

Proof. It suffices to prove (3.6) when f has support in a compact subset K of a local coordinate patch $Y \times [0,1)$ at the boundary, $Y \subset \mathbb{R}^{d-1}$. Let $\chi \in C_0^\infty(T \times [0,1))$ have $\chi \equiv 1$ in a neighborhood of K, let $k \geq s - m$. Then, with the notation $\xi = (\xi_1, \xi')$, and

$$\|u\|_{H_h^{(m,s)}}^2 := (2\pi h)^{-d} \int |\mathcal{F}_h(u)(\xi)|^2 \langle\xi\rangle^{2m} \langle\xi'\rangle^{2s} d\xi,$$

where $\mathcal{F}_h$ is the semiclassical Fourier transform, we have

$$\|f^0\|_{H_h^{(-m+s-k,k)}} \leq \|f^0\|_{H_h^{(0,s-m)}} \leq \|f\|_{H_h^{s-m}}.$$

We can write

$$(hD)^\alpha \chi T = \sum_{|\beta| \leq |\alpha|} T_\beta (hD)^\beta$$

where T_β is a pseudodifferential operator of order $-m$ and $\beta_n = 0$ if $\alpha_n = 0$. Then, since $\|(hD)^\beta f^0\|_{H_h^{s-m-k}} \leq \|f^0\|_{H_h^{(s-m-k,k)}}$ if $|\beta| \leq k$ and $\beta_n = 0$, we have

$$\|(hD)^\alpha \chi T f^0\|_{H_h^{s-k}} \leq C\|f^0\|_{H_h^{(s-m-k,k)}}$$

if $\alpha_n = 0$ and $|\alpha| \leq k$. Thus,

$$\|\chi T f^0\|_{H_h^{(s-k,k)}} \leq C\|f\|_{H_h^{(s-m)}}.$$

But, $PTf^0 = f$ in Ω^o. Hence, by [14, Theorem B.2.9], or rather a rescaling of its proof, we have for $\psi \in C_0^\infty$ with $\psi \equiv 1$ in a neighborhood of K and $\chi \equiv 1$ in a neighborhood of supp ψ,

$$\|\psi T f^0\|_{H_h^s} \leq C(\|f\|_{H_h^{s-m}} + \|\chi T f\|_{H_h^{(s-k,k)}}) \leq C\|f\|_{H_h^{s-m}}.$$

Since $\|(1-\psi)Tf^0\|_{H_h^s} = O(h^\infty)\|f^0\|_{H_h^{s-m}}$, we have proved (3.6).

Now, to prove (3.7), we may assume that supp $U \subset K$. We have,

$$P^c U = \sum_0^{m-1} v_j \otimes (hD_n)^j \delta, \quad v_j = \sum_{j+l<m} P_{j+l+1} U_l h i^{-1}.$$

Then, since P_{j+l+1} is order $m-j-l-1$, we have

$$\sum \|v_j\|_{H_h^{s-m+j+\frac{1}{2}}} \le Ch \sum \|U_j\|_{H_h^{s-j-\frac{1}{2}}}.$$

The semiclassial Fourier transform of $v_j \otimes (hD_1)^j \delta$ is $\mathcal{F}_h(v_j)(\xi')\xi_1^j$, and, when $j < m$,

$$h^{-1} \int \xi_1^{2j}(1+|\xi'|^2+\xi_1^2)^{-m}d\xi_1 \le Ch^{-1}(1+|\xi'|^2)^{j-m+\frac{1}{2}}.$$

Thus, if $j < m$ and k is an integer, $k \ge \max(s,0)$.

$$\|v_j \otimes (hD_1)^j \delta\|_{H_h^{(-m+s-k,k)}} \le \|v_j \otimes (hD_1)^j\|_{H_h^{(-m,s)}} \le Ch^{-1}\|v_j\|_{H_h^{s-m+j+\frac{1}{2}}}.$$

Putting these together, we have

$$\|TP^c U\|_{H_h^{(s-k,k)}} \le C \sum_0^{m-1} \|U_j\|_{H_h^{s-j-\frac{1}{2}}}.$$

Then, because T is continuous from $H_h^{(t-m,k)}$ to $H_h^{(t,k)}$ for $k \ge 0$, we can commute T with x' derivatives and hence we can improve this estimate to (3.7) in the same way as above. $\qquad\square$

Proposition 1 now follows from the fact that R is an inverse for (3.1) modulo $O(h)$ errors and an application of a standard Neumann series argument that can be found, for example, in [24, Theorem C.3].

4. Construction of Quasimodes Via Boundary WKB method

In this section, we will prove part 3 of Theorem 1. Moreover, we do not assume that X is constant in the construction. In particular, we prove

Proposition 2. *Let X be a vector field and $\partial\Omega_+$ defined as in (1.3). Then, for each $x_0 \in \partial\Omega_+$, let $\nu(x_0)$ be the outward unit normal. Then, if $d \ge 2$, for each*

$$z \in \{\zeta \in \mathbb{C} : |X(x_0)|^{-2}(\operatorname{Im}\zeta)^2 < \operatorname{Re}\zeta\}, \quad z \ne \langle X(x_0), \nu(x_0)\rangle^2/4,$$

there exists $u \in C^\infty(\Omega)$ such that u is a quasimode for z with $ES_h(u) = x_0$. Moreover, if $\partial\Omega$ and X are real analytic near x_0, then $Pu = O(e^{-\delta/h})$.

If $d = 1$, then for each $x_0 \in \partial\Omega_+$ and each

$$z \in \{\zeta \in \mathbb{C} : |X(x_0)|^{-2}(\operatorname{Im}\zeta)^2 < \operatorname{Re}\zeta\}, \quad z \notin \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta = 0 \text{ and } \operatorname{Re}\zeta \ge \langle X(x_0), \nu(x_0)\rangle^2/4\},$$

there exists $u \in C^\infty(\Omega)$ with the same properties as above.

Remark: We demonstrate the construction in dimension $d \geq 2$. The extra restriction in $d = 1$ comes from the fact that $\partial\Omega$ is a discrete set of points and hence functions on $\partial\Omega$ are determined by their values at these points. In particular, since we cannot choose $d\phi_0$ for ϕ_0 in equation (4.3), we must restrict the z for which we make the construction.

We wish to construct a solution to (1.2) that concentrates at a point in $\partial\Omega_+$. Let $x_0 \in \partial\Omega_+$ and assume for simplicity that $|X(x_0)| = 1$ and without loss that $X(x_0) = e_1$. We also assume $\operatorname{Im} z \neq \frac{\nu_1^2}{4}$ for technical reasons. To accomplish the construction, we postulate that u has the form

$$u = \chi v, \quad v = (ae^{i\varphi_1/h} - be^{i\varphi_2/h})$$

where

$$(4.1) \qquad a = \sum_{n=0}^{N} a_n h^n \quad b = \sum_{n=0}^{N} b_n h^n.$$

Then, let $\Gamma \subset \overline{\Omega}$ be a small neighborhood of x_0 to be determined later and $U \subset \overline{\Omega}$ be a small neighborhood of Γ. We solve

$$(4.2) \qquad \begin{cases} Pv = O(h^{N+2}) & \text{in } U, \\ v|_\Gamma = 0. \end{cases}$$

To do this, we need to solve

$$(4.3) \qquad \begin{cases} p(x, \partial\varphi_N) = O(|x - x_0|^{2N+4}) & \text{in } U, \\ \varphi|_\Gamma = \phi_0, \end{cases}$$

with $\phi_0(x_0) = 0$,

$$(4.4) \qquad d\phi_0(x_0) = \lambda \begin{cases} e_1'^\flat & |\nu_1| < 1, \\ e_2^\flat & \nu_1 = 1, \ d \geq 2 \end{cases} \in T^*\Gamma,$$

and $\operatorname{Im} d^2\phi_0(x_0) > 0$, where $e_1' = (e_1 - \langle e_1, \nu(x_0)\rangle\nu)/X'$, $X' = \sqrt{1 - \nu_1^2}$, and $\nu_1 = \langle e_1, \nu(x_0)\rangle$. (We choose ϕ in this way to get localization along the boundary.) We also need

$$(4.5) \qquad \begin{cases} -i\Delta\varphi_N a_n + 2i\langle\partial\varphi_N, \partial a_n\rangle + \langle X, \partial a_n\rangle = \Delta a_{n-1} + O(|x - x_0|^{2N+4}) & \text{in } U, \\ a_0|_\Gamma = 1 \quad a_n|_\Gamma = 0 \text{ for } n > 0, \end{cases} .$$

(4.3) has two solutions (φ_1 and φ_2) and a_n corresponds to φ_1 and b_n to φ_2.

First, we consider (4.3). To solve this equation, we construct a complex Lagrangian submanifold as in [4, Theorem 1.2']. Note that with the choice of $d\phi(x_0)$ as in (4.4),

$$d\varphi(x_0) = (f + ig)\nu^\flat(x_0) + d\phi(x_0).$$

This gives rise to

$$f^2 - g^2 - g\nu_1 - \operatorname{Re} z + \lambda^2 = 0$$

and

$$f(2g + \nu_1) - \operatorname{Im} z + X'\lambda = 0.$$

Hence

$$(g^2 + g\nu_1 + \operatorname{Re} z - \lambda^2)(2g + \nu_1)^2 = (\operatorname{Im} z - X'\lambda)^2.$$

Letting $c = g + \frac{\nu_1}{2}$, we have

$$c^4 + \left(\operatorname{Re} z - \lambda^2 - \frac{\nu_1^2}{4}\right) c^2 - \frac{1}{4}(\operatorname{Im} z - X'\lambda)^2 = 0.$$

Which gives

$$c = \pm \frac{1}{\sqrt{8}} \left(\sqrt{\nu_1^2 - 4(\operatorname{Re} z - \lambda^2) + \sqrt{(\nu_1^2 - 4(\operatorname{Re} z - \lambda^2))^2 + 16(\operatorname{Im} z - X'\lambda)^2}}\right)$$

(note that we take the positive root inside so that the result is real)

In order to complete the construction, we need $g < 0$. That is, we require

$$
\begin{aligned}
\frac{\nu_1}{2} > |c| \quad \Leftrightarrow \quad & 2\nu_1^2 > \nu_1^2 - 4(\operatorname{Re} z - \lambda^2) + \sqrt{(\nu_1^2 - 4(\operatorname{Re} z - \lambda^2))^2 + 16(\operatorname{Im} z - X'\lambda)^2} \\
\Leftrightarrow \quad & (\nu_1^2 + 4(\operatorname{Re} z - \lambda^2))^2 > (\nu_1^2 - 4(\operatorname{Re} z - \lambda^2))^2 + 16(\operatorname{Im} z - X'\lambda)^2 \\
\Leftrightarrow \quad & (\operatorname{Re} z - \lambda^2)\, \nu_1^2 > (\operatorname{Im} z - X'\lambda)^2 \\
\Leftrightarrow \quad & \operatorname{Re} z > \frac{(\operatorname{Im} z)^2(1 - X'^2) + (\lambda - \operatorname{Im} z X')^2}{\nu_1^2} = (\operatorname{Im} z)^2 + \frac{(\lambda - \operatorname{Im} z X')^2}{\nu_1^2}.
\end{aligned}
$$

We also need $|c| > 0$ so that φ_1 and φ_2 are distinct. Thus, letting $|a| < 1$, we choose

$$
\lambda = \begin{cases}
\operatorname{Im} z X' & \operatorname{Im} z \neq 0, \\
\nu_1 \sqrt{\operatorname{Re} z}\, a & \operatorname{Im} z = 0, \nu_1 \neq 1, \\
0 & \operatorname{Im} z = 0, \operatorname{Re} z < \frac{1}{4}, \nu_1 = 1, \\
\sqrt{\operatorname{Re} z - \frac{1}{8}} & \operatorname{Im} z = 0, \operatorname{Re} z \geq \frac{1}{4}, \nu_1 = 1.
\end{cases}
$$

Remark: In dimension 1, we are forced to choose $\lambda = 0$, however, in dimension 1, $\nu_1 = 1$, so we have $|c| > 0$ when $\operatorname{Re} z < \frac{1}{4} = \frac{\nu_1^2}{4}$.

With this choice for λ, we have $0 < |c| < \frac{\nu_1}{2}$ if and only if $(\operatorname{Im} z)^2 < \operatorname{Re} z$. Hence, v decays exponentially in the $-x_1$ direction if $(\operatorname{Im} z)^2 < \operatorname{Re} z$. This decay allows us to localize our construction near the boundary.

Remark: $(\operatorname{Im} z)^2 < \operatorname{Re} z$, corresponds precisely with (3.3).

Now that we have f and g, and λ, we need to solve (4.3) on the rest of Γ and in the interior of U. To do this, we first assume that Γ and p are real analytic and solve the equations exactly. Let γ be the coordinate change to semigeodesic coordinates for Γ. Extend γ analytically to a neighborhood of Γ in $\mathbb{C}^{d-1}$. Then, define κ, the lift of γ, by $(z, \eta) \mapsto (\gamma(z), (\partial \gamma^{-1})^T \eta)$. Next, choose $\phi_1(y)$ real analytic in a neighborhood of $0 \in \mathbb{R}^{d-1}$ with $\phi_1(0) = 0$, $d\phi_1(0)$ as in (4.4), and $\operatorname{Im} d^2\phi_1(0) > 0$. Then, extend ϕ_1 to y in a neighborhood of the origin in $\mathbb{C}^{d-1}$. Next, let

$$\Lambda_0 := \{(0, y, \xi_1(y), d_y\phi_1(y)) : \kappa_* p(0, y, \xi_1, d_y\phi_1(y)) = 0, (0, y) \in \gamma(\Gamma)\}$$

where $\xi_1(y)$ is well defined since $\xi_1(0) = f + ig$ and, for $z \neq \frac{\nu_1^2}{4}$, $\partial_{\xi_1} \kappa_* p(x_0, f + ig, d\phi(x_0)) \neq 0$. Observe also that Λ_0 is isotropic with respect to the complex symplectic form.

Finally, let Φ_t be the complex flow of $\kappa_* p$ which exists by the Cauchy-Kovalevskaya Theorem ([6, Section 4.6]). Then,

$$\Lambda := \cup_{|t| < \epsilon} \Phi_t(\Lambda_0)$$

is Lagrangian. Hence it has generating function $\tilde{\varphi}$ such that $\varphi = \tilde{\varphi} \circ \gamma$ solves (4.3) and has $\varphi|_\Lambda = \phi_0 := \phi_1 \circ \gamma$. Therefore, there exists $\varphi_1 \neq \varphi_2$, solutions to (4.3).

Remark: Note that the two distinct solutions φ_1 and φ_2 come from the two solutions to $\xi_1(x_0)$.

Next, we solve (4.5). To do this, note that φ_1 and φ_2 from above are analytic. Hence, since Γ and (4.5) are analytic, we may apply the Cauchy-Kovalevskaya Theorem as above to find a_n and b_n.

If Γ and p are analytic, it is classical [21, Theorem 9.3] that the solutions a_n and b_n have $\max(|a_n|, |b_n|) \leq C^n n^n$. Thus, the error contributed by truncation at $N = 1/Ch$ is exponential.

Suppose that Γ and p are not analytic. Then, let γ be the coordinate change to semigeodesic coordinates for Γ. Define the lift κ of γ and choose ϕ_1 as above. We now solve the equations (4.3) with $O(|x - x_0|^{2N+4})$ error. First, write

$$\kappa_* p = p_1 + O(|x - x_0|^{2N+4}) p'(\xi),$$

where p_1 is the Taylor polynomial for $\kappa_* p$ to order $2N + 4$. Next, apply the construction for analytic p from above to solve

$$\begin{cases} p_1(x, \partial\theta) = 0, \\ \theta|_{\gamma(\Gamma)} = \phi_1. \end{cases}$$

Then, observe that

$$\kappa_* p(x, \partial\theta) = O(|x - x_0|^{2N+4}) p'(\partial\theta) = O(|x - x_0|^{2N+4}).$$

Hence, we have that $\varphi_N = \theta \circ \gamma$ solves (4.3).

Now, using the solution φ_N just obtained, we solve the amplitude equations (4.5) with $O(|x - x_0|^{2N+4})$ errors. As with the phase, we start by changing to semigeodesic coordinates. Write the equation for a_n in the new coordinates as

$$\begin{cases} \langle \rho, \partial a_n' \rangle - \psi a_n' = f, \\ a_0'|_{\gamma(\Gamma)} = 1 \quad a_n'|_{\gamma(\Gamma)} = 0 \text{ for } n > 0. \end{cases}$$

Then, writing the Taylor polynomials to order $2N+4$ for ρ, ψ, and f as ρ_1, ψ_1, and f_1 respectively, we solve

$$\begin{cases} \langle \rho_1, \partial a_n' \rangle - \psi_1 a_n' = f_1, \\ a_0'|_{\gamma(\Gamma)} = 1 \quad a_n'|_{\gamma(\Gamma)} = 0 \text{ for } n > 0, \end{cases}$$

using the analytic construction above. Then, just as in the solution of (4.3), $a_n := a_n' \circ \gamma$ solves (4.5).

Now, to complete the construction, let $V \Subset U$ be a neighborhood of Γ. Then, let $\chi \in C^\infty(\Omega)$ with $\chi \equiv 1$ on V and $\chi \equiv 0$ on $\Omega \setminus U$. For convenience, we make another change of coordinates so $x_0 \mapsto 0$ and that $\nu(x_0) = e_1$. Then, $\operatorname{Im} \partial_{x_1} \varphi_i(0) < 0$ for $i = 1, 2$, and $\operatorname{Im} d^2\phi(0) > 0$. Together, these imply that $\operatorname{Im} \varphi > 0$ on $\operatorname{supp} \partial\chi \cap \Gamma$. Hence, we have for U small enough but independent of h,

$$Pu = \chi Pv + [P, \chi]v = O(|x|^{2N+4})(e^{i\varphi_1/h} + e^{i\varphi_2/h}) + O(h^{N+2}) + O(e^{-\epsilon/h}).$$

Now, observe that $a|_\Gamma = b|_\Gamma$, $\varphi_1|_\Gamma = \varphi_2|_\Gamma = \phi_0$ and

$$\varphi_i = \phi_0(x') + c_i x_1 + O(x_1|x'|) + O(x_1^2),$$

$c_1 \neq c_2$. Hence, since $\operatorname{Im}\phi_0(x') \geq c|x'|^2$, and $\operatorname{Im} c_i < 0$

$$O(|x|^{2N+4})(e^{i\varphi_1/h} + e^{i\varphi_2/h}) = O(h^{N+2})$$

and v solves (4.2).

Note also that if Γ and P were analytic, then the equations (4.3) and (4.5) can be solved exactly with $\max(|a_n|, |b_n|) < C^n n^n$. Hence, truncating the sums (4.1) at $N = 1/eCh$, we have

$$Pu = \chi Pv + [P,\chi]v = C^N N^N h^{N+1} + O(e^{-\epsilon/h}) = (CN)^N (CN)^{-N} e^{-N} + O(e^{-\epsilon/h})$$
$$= e^{-c/h} + O(e^{-\epsilon/h}) = O(e^{-\epsilon/h}).$$

Our last task is to show that $\|u\|_{L^2} \geq Ch^{\frac{d+3}{4}}$. To see this, we calculate, shrinking U and V if necessary, and letting $u = \chi v$,

$$
\begin{aligned}
\|u\|_{L^2}^2 &= \int_V \left|(1 + O(x))\left(e^{i\varphi_1/h} - e^{i\varphi_2/h}\right)\right|^2 \\
&\geq c\int_{\gamma(V)} \left|e^{i(\phi_0(x')+O(|x'|x_1)+O(x_1^2))/h}\left(e^{ic_1x_1/h} - e^{ic_2x_1/h}\right)\right|^2 dx_1 dx' \\
&\geq c\int_{\gamma(V)} e^{-c|x'|^2/h} e^{-c_0 x_1/h} x_1 \left(1 + O(e^{-\delta x_1/h})\right) dx_1 dx' \geq ch^{\frac{d+3}{2}}
\end{aligned}
$$

(Also note that by the same argument $\|u\|_{L^2}^2 \leq Ch^{\frac{d+3}{2}}$.) To finish the construction of u, we simply rescale u so that it has $\|u\|_{L^2} = 1$ and invoke Borel's Theorem (see, for example [24, Theorem 4.15]).

5. Propagation of Semiclassical Wavefront Sets

We first examine the case where $P = hD_{x_1} + ihD_{x_2} = -ih\partial_{\bar{z}}$. We make the following definition in the spirit of Duistermaat and Hörmander [5, Section 7]

Definition 4.

$$s_u^0(x) := \sup\{t \in \mathbb{R} : h^{-t}u = O_{L^2}(1) \text{ in a neighborhood of } x\}$$

We will need the following lemma

Lemma 4. *Let $u \in H_h^1$ with $hD_{\bar{z}}u = f$. Then, $s^* = \min(s_u^0, s)$ is superharmonic if s is superharmonic and $s_f^0 - 1 \geq s$.*

Proof. Let $q(z)$ be harmonic function in $\mathbb{C}$ such that $s^*(z,0) > q(z)$ for $|z| = r$. Then, we need to show that $s^*(z) > q(z)$ for $|z| \leq r$. The fact that $s(z,0) > q(z)$ for $|z| \leq r$ follows from the superharmonicity of s. Therefore, we only need to show the inequality for s_u^0.

To do this, let $\chi_1 \in C_0^\infty(\mathbb{R}^{d-2})$ have support in a small neighborhood of 0, $\chi_1(0) = 1$, and $\chi_2 \in C_0^\infty(\mathbb{C})$ be 1 for $|z| \leq r$ and 0 outside a neighborhood so small that

$$\|h^{-q(z)}u\|_{L^2(\text{supp } \chi_1 \partial \chi_2)} = O(1).$$

This is possible since $s^* > q(z)$ for $|z| = r$ implies $s_u^0 > q(z)$ for $|z| = r$. Then, define $v := \chi_1 \chi_2 u$. We have

$$hD_{\bar{z}}v = \chi_1 \chi_2 f + \chi_1 u h D_{\bar{z}}(\chi_2) =: g$$

with $s_g^0(z) > q(z) + 1$.

Next, let $F(z)$ be analytic with $\text{Re}\, F(z) = q(z)$ and define $Q(z) := h^{-F(z)}$. Using this, we have $D_{\bar{z}}(Qv) = h^{-1}Qg$ and, since $s_g^0 > q + 1$, $h^{-1}Qg = O_{L^2}(1)$. Then, applying ∂_z, we have

$$\Delta_{x_1,x_2}(Qv) = CD_z D_{\bar{z}}(Qv) = Ch^{-1}D_z Qg$$

and hence, shrinking r if necessary (note that this is valid since superharmonicity is a local property), Qv solves,

$$\begin{cases} \Delta_{x_1,x_2}(Qv) = Ch^{-1}D_z(Qg) & \text{in } B(0,1) \subset \mathbb{R}^2, \\ Qv = 0 & \text{in } \partial B(0,1). \end{cases}$$

Therefore, by the estimate

$$\|u\|_{L^2} \leq C\|\Delta u\|_{H^{-1}},$$

we have

$$\|Qv\|_{L^2_{x_1,x_2}} \leq C\|h^{-1}D_z Qg\|_{H^{-1}_{x_1,x_2}}.$$

But, since $h^{-1}Qg = O_{L^2}(1)$, $\|h^{-1}D_z Qg\|_{H^{-1}_{x_1,x_2}} = O(1)$ for almost every $x' \in \text{supp } \chi_1$, the same is true for Qv. Thus, since $v \equiv u$ in a neighborhood of 0, and $|Q| = |h^{-q(z)}|$, $s_u^0 > q(z)$ for $|z| \leq r$. $\qquad\square$

Definition 5. *We say that an operator T quantizes κ if $T : L^2 \to L^2$ and for all $a \in S(m)$, we have*

$$T^{-1}a^w(x, hD)T = b^w(x, hD)$$

for a symbol $b \in S(m)$ satisfying

$$b|_{U_0} := \kappa^*(a|_{U_1}) + O_S(h).$$

(The definition of the symbol class $S(m)$ can be found in [24, Chapter 4].)

To convert from P as in (1.1) to the case of $P = hD_{\bar{z}}$ we need the following lemma similar to [24, Theorem 12.6] which we include for completeness.

Lemma 5. *Suppose $P = p^w$ is of semiclassical principal type with $p_0(0,0) = 0$ and $\{\text{Re}\, p, \text{Im}\, p\} = 0$ with $\partial\, \text{Re}\, p$ and $\partial\, \text{Im}\, p$ linearly independent. Then there exists a local canonical transformation κ defined near $(0,0)$ such that*

$$\kappa^* p_0 = \xi_1 + i\xi_2$$

and an operator $T : L^2 \to L^2$ quantizing κ in the sense of definition 5 such that T^{-1} exists microlocally near $((0,0),(0,0))$ and

$$TPT^{-1} = hD_{x_1} + ihD_{x_2} \quad \text{microlocally near } ((0,0),(0,0)).$$

Proof. Let $p_1 = \operatorname{Re} p_0$ and $p_2 = \operatorname{Im} p_0$. Then, by a variant of Darboux's Theorem (see, e.g. [24, Theorem 12.1]), there exists κ a symplectomorphism, locally defined near $(0,0)$, such that $\kappa(0,0) = (0,0)$ and

$$\kappa^* p_1 = \xi_1 \quad \kappa^* p_2 = \xi_2.$$

Then, by [24, Theorem 11.6] shrinking the domain of definition for κ if necessary, there exists a unitary T_0 quantizing κ such that

$$T_0 P T_0^{-1} = hD_{x_1} + ihD_{x_2} + E \quad \text{microlocally near } (0,0),$$

where $E = e^w$ for $e \in hS$.

Next, we find $a \in S$ elliptic at $(0,0)$ such that

$$hD_{x_1} + ihD_{x_2} + E = A(hD_{x_1} + ihD_{x_2})A^{-1} \text{ microlocally near } (0,0),$$

where $A = a^w$ i.e.

$$[hD_{x_1} + ihD_{x_2}, A] + EA = 0 \text{ microlocally near } (0,0).$$

Since $P = p_0^w + h p_1^w + ...$, we have $E = e^w$ for $e = h e_1 + h^2 e_2 + ...$ We use the Cauchy formula to solve the equation

$$\frac{1}{i}\{\xi_1 + i\xi_2, a_0\} + e_0 a_0 = 0$$

near $(0,0)$ for $a_0 \in S$ with $a_0(0,0) \neq 0$. Then, defining $A_0 := a_0^w$, we have

$$[hD_{x_1} + ihD_{x_2}, A_0] + EA_0 = r_0^w$$

for $r_0 \in h^2 S$. To complete the proof, we proceed inductively to obtain $A_k = a_k^w$ for $a_k \in S$, solving

$$[hD_{x_1} + ihD_{x_2}, A_0 + ... + h^N A_N] = E(A_0 + ... + h^N A_N) = r_N^w$$

where $r_N \in h^{N+2} S$, using the Cauchy formula at each stage. Then, we invoke Borel's Theorem (see, for example [24, Theorem 4.15]) to find A and let $T = A^{-1} T_0$. $\qquad\square$

Now, define

Definition 6.

$$S_u^0(x,\xi) := \sup\{t \in \mathbb{R} : \exists \; U, V \; open \; x \in U, \; \xi \in V \; s.t.$$
$$\forall \; \chi_1 \in C_0^\infty(U), \chi_2 \in C_0^\infty(V), h^{-t}(\chi_1)\chi_2(hD)u = O_{L^2}(1)\}$$

We prove that definition 6 is equivalent to the following definition

Definition 7.

$$S_u(x,\xi) := \sup\{t \in \mathbb{R} : \; there \; exists \; U, \; (x,\xi) \in U, \; s.t. \; \forall \; \chi \in C_0^\infty(U) \; h^{-t}\chi^w(x,hD)u = O_{L^2}(1)\}.$$

The proof follows [24, Theorem 8.13], but we reproduce it in this setting for the convenience of the reader.

Lemma 6. *Suppose that there exist U and V as in definition 6. Then, there exists W open, $(x,\xi) \in W$ such that for $\chi \in C_0^\infty(V)$, $\chi^w u = O_{L^2}(h^t)$.*

Proof. Let $a = \chi_1(x)\chi_2(\xi)$ as in definition 6. Then, there exists $\chi \in C^\infty$ supported near (x_0, ξ_0) such that

$$|\chi(x, \xi)(a(x, \xi) - a(x_0, \xi_0)) + a(x_0, \xi_0)| \geq \gamma > 0.$$

Hence, by [24, Theorem 4.29], there exists $c \in S$ such that for h small,

$$c^w(\chi^w a^w + a(x_0, \xi_0)(I - \chi^w)) = I.$$

Next, observe that

$$b^w u = b^w c^w \chi^w a^w u + a(x_0, \xi_0) b^w c^w (I - \chi^w) u.$$

Now, the first term on the right is $O(h^t)$ since $a^w u = O(h^t)$. Also, if the support of b is sufficiently near (x_0, ξ_0), supp $b \cap$ supp $(1 - \chi) = \emptyset$ and hence the second term is $O(h^\infty)$. This proves the claim. $\qquad\square$

Remark: Note that $S_u(x, \xi) = \infty$ if and only if $(x, \xi) \notin WF_h(u)$.

Lemma 6 shows that $S_u(x, \xi) = S_u^0(x, \xi)$. It will be convenient to change between both definitions in the proof of the following proposition.

Proposition 3. *Let $u \in H^m$, $Pu = f$, and let $S_f \geq s + 1$, $\Omega \subset N$ where*

$$N := \left\{ (x, \xi) \in T\mathbb{R}^d : p(x, \xi) = 0, \ \{p, \bar{p}\} = 0, \ H_{\operatorname{Re} p} \text{ and } H_{\operatorname{Im} p} \text{ are independent} \right\}.$$

Then, it follows that $\min(S_u, s)$ is superharmonic in Ω if s is superharmonic in Ω, and that $\min(S_u - s, 0)$ is superharmonic in Ω if s is sub harmonic in Ω (with respect to H_p). In particular, S_u is superharmonic in Ω if $\Omega \cap WF_h(f) = \emptyset$.

Proof. First, we consider $hD_{\bar{z}}$. We prove that Lemma 4 remains valid with s_u^0 and s_f^0 replaced by S_u^0 and S_f^0. Let $\chi \in C_0^\infty(\mathbb{R}^d)$. Then, $hD_{\bar{z}}(\chi(hD)u) = \chi(hD)f$. Then, taking χ_j with $\chi_j(\xi_0) = 1$, vanishing outside V_j, $V_j \downarrow \{\xi_0\}$, and denoting $u_j = \chi_j(hD)u$, we have $s_{u_j}^0(x) \uparrow S_u^0(x, \xi_0)$. So, the superharmonicity of

$$\min(s_{u_j}^0, s_j), \text{ where } s_j(x) = \inf_{\xi \in V_j} s(x, \xi),$$

gives that $\min(S_u^0, s)$ is superharmonic and proves the first part of the Proposition for $hD_{\bar{z}}$.

To prove the second, note that it is equivalent to the first if s is harmonic. Thus, the second part follows if s is the supremum of a family of harmonic functions. If $s \in C^2$ is strictly subharmonic, then $s(z, x', \xi) \geq q(z, x', \xi)$ in a neighborhood of (w, x', ξ) with equality at (w, x', ξ) when q is the harmonic function

$$q(z, x', \xi) = s(w, x', \xi) + \operatorname{Re}(2(z - w)\partial s(w, x', \xi)/\partial w + (z - w)^2 \partial^2 s(w, x', \xi)\partial w^2).$$

Then, the local character of superharmonicity proves the second statement when s is strictly subharmonic and the general case follows by approximation of s with such functions.

To pass from $hD_{\bar{z}}$ to P, we need the following lemma ([5, Lemma 7.2.3]).

Lemma 7. *If $(x_0, \xi_0) \in N$ one can find a C^∞ function a of degree $1 - m$ with $a(x_0, \xi_0) \neq 0$ such that $\{q, \bar{q}\} = 0$ in a neighborhood of (x_0, ξ_0) if $q = ap$.*

Now, by Lemma 5, there exists T microlocally quantizing κ such that $\kappa^*(\operatorname{Re} ap) = \xi_1$ and $\kappa^*(\operatorname{Im} ap) = \xi_2$, so that

$$Ta^w P T^{-1} = hD_{x_1} + ihD_{x_2}$$

microlocally near $((x_0, \xi_0), (0,0))$. Then,

$$S_{Tu} \circ \kappa(x, \xi) = S_u(x, \xi),$$

for $(x, \xi) \in V$ a small neighborhood of (x_0, ξ_0). This follows from the fact that, by Definition 5,

$$h^{-t} T^{-1} T \chi^w T^{-1} Tu = h^{-t} T^{-1} b^w Tu$$

where $b|_{U_0} = \kappa^*(\chi|_{U_1}) + O_S(h)$ and T^{-1} is uniformly bounded on L^2. But $S^0_{Ta^w Pu} = S_{a^w f} = S_f = S^0_f \geq s + 1$. Hence, Proposition 3 follows from the case with $hD_{\bar{z}}$. $\qquad \square$

We need the following elementary lemma to prove Corollary 1.

Lemma 8. *Suppose u solves (1.2) and P has symbol $p(x, \xi)$. Then,*

$$WF_h(u) \cap (\Omega^o \times \mathbb{R}^d) \subset p^{-1}(0) \cap (\Omega^0 \times \mathbb{R}^d).$$

Proof. Let $x_0 \in \Omega^o$ and $(x_0, \xi_0) \notin p^{-1}(0)$. Then, let $\chi_1 \in C_0^\infty(\mathbb{R}^d)$ have support near x_0 and $\chi_2 \in C_0^\infty(\mathbb{R}^d)$ have support near ξ_0. Then we have

$$\chi_2^w \chi_1^w p^w u = O(h^\infty).$$

But, $\chi_2^w \chi_1^w = c^w$, $c \in \mathcal{S}$ with $|c(x_0, \xi_0)| > 0$. Similarly, $c^w p^w = q^w$ for $q \in \mathcal{S}$ with $|q(x_0, \xi_0)| > 0$. Hence, $(x_0, \xi_0) \notin WF_h(u)$. $\qquad \square$

Putting Proposition 3 together with Lemma 8, we have the following corollary

Corollary 1. *Let P as in (1.1), $u \in H_h^2$ with $\operatorname{Re} z > (\operatorname{Im} z)^2 |X|^{-2}$. Let $u \in H_h^2$ with $(P - z)u = O_{L^2}(h^\infty)$ Then, $WF_h(u) \cap (\Omega^o \times \operatorname{Re}^d)$ is invariant under the leaves generated by $H_{\operatorname{Im} p}$ and $H_{\operatorname{Re} p}$.*

Proof. By Lemma 8,

$$WF_h(u) \cap (\Omega^o \times \operatorname{Re}^d) \subset p^{-1}(0) \cap (\Omega^o \times \mathbb{R}^d).$$

We also have that $\{p, \bar{p}\} = 0$, and for $\operatorname{Re} z > (\operatorname{Im} z)^2 |X|^{-2}$, $H_{\operatorname{Re} p}$ and $H_{\operatorname{Im} P}$ are independent on all of $p^{-1}(0)$. Now, let $K_n \Subset \Omega^o$ and $K_n \uparrow \Omega^o$. Then, let $\chi_n \in C_0^\infty(\Omega^o)$ and $\chi_n \equiv 1$ on K_n. Then, applying Proposition 3, to $\chi_n u$, we have that $WF_h(\chi_n u) \cap K_n \times \mathbb{R}^d$ is invariant under the leaves generated by $H_{\operatorname{Im} p}$ and $H_{\operatorname{Re} p}$. But, this is true for all n, so, letting $n \to \infty$, we obtain the result. $\qquad \square$

6. A Carleman Type Estimate

We now prove a Carleman type estimate for (P, Ω). This will be used in the following sections to restrict the essential support of quasimodes.

Observe that for $\varphi \in C^\infty$, we have

$$(6.1) \qquad P_\varphi := e^{\varphi/h} P e^{-\varphi/h} = \sum (hD_{x_j} + i\partial_{x_j}\varphi)^2 - \langle X, \partial\varphi \rangle + i\langle X, hD \rangle - z$$

with Weyl symbol

$$(6.2) \qquad p_\varphi(x, \xi) = |\xi|^2 - \langle X + \partial\varphi, \partial\varphi \rangle + i\langle X + 2\partial\varphi, \xi \rangle - z.$$

Then, $P_\varphi = A + iB$ where A and B are formally self adjoint and have

$$A = (hD)^2 - \langle X + \partial\varphi, \partial\varphi \rangle - \operatorname{Re} z, \quad B = \langle X, hD \rangle + \sum_j (\partial_{x_j}\varphi \circ hD_{x_j} + hD_{x_j} \circ \partial_{x_j}\varphi) - \operatorname{Im} z$$

with symbols

$$a = |\xi|^2 - \langle X + \partial\varphi, \partial\varphi \rangle - \operatorname{Re} z, \quad b = \langle X + 2\partial\varphi, \xi \rangle - \operatorname{Im} z.$$

Also, note that

$$[A, B] = hi^{-1} \left(4\langle \partial^2\varphi hD, hD \rangle + \langle \partial^2\varphi(X + 2\partial\varphi), X + 2\partial\varphi \rangle + 4hi^{-1}\langle \partial\Delta\varphi, hD \rangle + h^2 i^{-2}\Delta\Delta\varphi \right).$$

Next, let $u \in C^\infty(\Omega)$ with $u|_{\partial\Omega} = 0$, $Pu = v$, $u_1 := e^{\varphi/h}u$, and $v_1 = e^{\varphi/h}v$. Then, we compute

$$\begin{aligned}
\|v_1\|^2 &= ((A + iB)u_1, (A + iB)u_1) \\
&= \|Au_1\|^2 + \|Bu_1\|^2 + i\left[(Bu_1, Au_1) - (Au_1, Bu_1)\right]
\end{aligned}$$

Now, observe that, since B is a first order differential operator that is formally self adjoint, and $u|_{\partial\Omega} = 0$,

(6.3) $(Au_1, Bu_1) = (BAu_1, u_1).$

Next,

(6.4) $(Bu_1, Au_1) = (ABu_1, u_1) - h^2 (Bu_1, \partial_\nu u_1)_{\partial\Omega}.$

But, on $\partial\Omega$

$$B = \frac{h}{i}\langle 2\partial\varphi + X, \nu \rangle \partial_\nu + B'$$

where B' acts along $\partial\Omega$. Hence,

$$(Bu_1, \partial_\nu u_1)_{\partial\Omega} = \frac{h}{i} \left(\langle 2\partial\varphi + X, \nu \rangle \partial_\nu u_1, \partial_\nu u_1 \right)_{\partial\Omega}$$

and we have

(6.5) $\|v_1\|^2 = \|Au_1\|^2 + \|Bu_1\|^2 + i\left([A, B]u_1, u_1\right) - h^3 \left(\langle 2\partial\varphi + X, \nu \rangle \partial_\nu u_1, \partial\nu u_1\right)_{\partial\Omega}.$

Next, we compute

$$\{a, b\} = 4\langle \partial^2\varphi \xi, \xi \rangle + \langle \partial^2\varphi(X + 2\partial\varphi), X + 2\partial\varphi \rangle.$$

Thus, choosing $\varphi = \epsilon\psi$ with $\partial^2\psi$ positive definite, we have

$$\{a, b\} \geq C\epsilon|\xi|^2 + C\epsilon|X + 2\epsilon\partial\psi|^2 \geq C\epsilon(|\xi|^2 + |X|^2) + O(\epsilon^2).$$

Now, $i[A, B] = h\{a, b\}^w + \epsilon h^2 r^w$, where $r = -4\langle \partial\Delta\psi, \xi \rangle + hi^{-1}\Delta\Delta\psi$ is order 1. Hence, for $\delta > 0$ small enough and independent of h, h small enough, and $0 < \epsilon < \delta$ (here ϵ may depend on h), we have

$$i[A, B] = Ch\epsilon(-h^2(\partial^2\psi)^{ij}\partial^2_{x_i x_j} + f(x)) + \epsilon O_{H^1_h \to L^2}(h^2)$$

where $f \geq C > 0$ and $\partial^2\psi > C \geq 0$. Hence, by an integration by parts,

$$i\left([A, B]u_1, u_1\right) \geq Ch\epsilon(\|hDu_1\|^2 + \|u_1\|^2).$$

Combining this with (6.5), noting that, on $\partial\Omega$, $\partial_\nu u_1 = e^{\frac{\epsilon\psi}{h}}\partial_\nu u$, and, letting Γ_+ and $\partial\Omega_-$ be as in (1.3), we have,

$$-h^3\left(\langle 2\epsilon\partial\psi + X, \nu\rangle e^{\frac{\epsilon\psi}{h}}\partial\nu u, e^{\frac{\epsilon\psi}{h}}\partial\nu u\right)_{\partial\Omega_-} + Ch\epsilon\|e^{\frac{\epsilon\psi}{h}}u\|^2_{H^1_h}$$
$$\leq \|e^{\frac{\epsilon\psi}{h}}Pu\|^2 + h^3\left(\langle 2\epsilon\partial\psi + X, \nu\rangle e^{\frac{\epsilon\psi}{h}}\partial\nu u, e^{\frac{\epsilon\psi}{h}}\partial\nu u\right)_{\Gamma_+}$$

Now, note that a similar proof goes through if

$$(6.6) \qquad\qquad\qquad \psi(x) = \psi_1(\langle x, X\rangle) + \psi_2(x)$$

where ψ_1 has $\langle\partial^2\psi_1 X, X\rangle > c|X|^2$ and ψ_2 is linear. After this observation, we obtain the following lemma,

Lemma 9. *Let $u \in C^\infty(\Omega)$ $u|_{\partial\Omega} = 0$, $\psi \in C^\infty$ either have*

(1) *ψ is locally strictly convex ($\partial^2\psi$ is positive definite), or*
(2) *ψ is as in (6.6).*

Then, there exists $\delta > 0$ independent of h small enough such that for $0 < \epsilon \leq \delta$ (ϵ possibly depending on h), and $0 < h < h_0$, we have

$$(6.7) \quad -h^3\left(\langle 2\epsilon\partial\psi + X, \nu\rangle e^{\frac{\epsilon\psi}{h}}\partial\nu u, e^{\frac{\epsilon\psi}{h}}\partial\nu u\right)_{\partial\Omega_-} + Ch\epsilon\|e^{\frac{\epsilon\psi}{h}}u\|^2_W$$
$$\leq \|e^{\frac{\epsilon\psi}{h}}Pu\|^2_{L^2} + h^3\left(\langle 2\epsilon\partial\psi + X, \nu\rangle e^{\frac{\epsilon\psi}{h}}\partial\nu u, e^{\frac{\epsilon\psi}{h}}\partial\nu u\right)_{\Gamma_+}$$

where if ψ satisfies

(1) $\|\cdot\|_W = \|\cdot\|_{H^1_h}$
(2) $\|\cdot\|_W = \|\cdot\|_{L^2} + \|\langle X, hD\rangle \cdot\|_{L^2}$.

Lemma 9 easily extends to $u \in H^2_h$ with $u|_{\partial\Omega} = 0$.

7. Essential Support of Quasimodes

In this section, we prove part 2 of Theorem 1.

7.1. No quasimodes in the boundary of the pseudospectrum.
Let $z_0 \in \partial\Lambda(P, \Omega)$. We use a small weight to conjugate P as in (6.1) such that p_φ is elliptic. For simplicity, we again assume $X = e_1$ and hence $\mathrm{Re}\, z_0 = (\mathrm{Im}\, z_0)^2$. Using (6.2), let $\epsilon > 0$ and $\partial\varphi = -\epsilon X$. Then, using the fact that $X = e_1$, we have

$$p_\varphi(x, \xi) = |\xi|^2 + (1-\epsilon)\epsilon + i(1-2\epsilon)\xi_1 - z_0.$$

Then, $p = 0$ implies that

$$\xi_1 = \frac{\mathrm{Im}\, z_0}{1 - 2\epsilon} \quad |\xi'|^2 + \frac{1}{(1-2\epsilon)^2}\left((\mathrm{Im}\, z_0)^2 - \mathrm{Re}\, z_0 - \mathrm{Re}\, z_0(-4\epsilon + 4\epsilon^2)\right) + \epsilon(1-\epsilon) = 0.$$

But,

$$|\xi'|^2 + \frac{1}{(1-2\epsilon)^2}\left((\operatorname{Im} z_0)^2 - \operatorname{Re} z_0 - \operatorname{Re} z_0(-4\epsilon + 4\epsilon^2)\right) + \epsilon(1-\epsilon)$$
$$\geq \frac{1}{(1-2\epsilon)^2}(4\epsilon \operatorname{Re} z_0(1-\epsilon)) + (1-\epsilon)\epsilon > 0$$

for ϵ small enough. We now show that

$$|p_\varphi| \geq c\epsilon\langle\xi\rangle^2$$

for ϵ small enough. To see this, let $\xi_1 = (\operatorname{Im} z_0 + \gamma)/(1-2\epsilon)$. Then, choose $\gamma = \delta\epsilon$

$$\begin{aligned}
|p_\varphi(x,\xi)| &\geq |\xi'|^2 + \frac{1}{(1-2\epsilon)^2}(\gamma^2 + 2\gamma \operatorname{Im} z_0 + 4\epsilon \operatorname{Re} z_0 - 4\epsilon^2 \operatorname{Re} z_0) + \epsilon - \epsilon^2 \\
&\geq (1 + O(\epsilon))(\delta^2\epsilon^2 + 2\delta\epsilon \operatorname{Im} z_0 + 4\epsilon \operatorname{Re} z_0 - 4\epsilon^2 \operatorname{Re} z_0) + \epsilon + O(\epsilon^2) \\
&\geq \epsilon(2\delta \operatorname{Im} z_0 + 4 \operatorname{Re} z_0 + 1) - O(\epsilon^2) \geq c'\epsilon
\end{aligned}$$

for δ small enough independent of ϵ and ϵ small enough.

Therefore, by Proposition 1, if $u|_{\partial\Omega} = 0$, we have that

$$\epsilon\|e^{\frac{\epsilon\psi}{h}}u\|_{H_h^2} \leq C\|e^{\frac{\epsilon\psi}{h}}Pu\|_{L^2}.$$

Thus, if u is a quasimode for z_0, choosing $\epsilon = h\log h^{-1}$,

$$\|u\|_{L^2} \leq C(h^N \log h^{-1})^{-1}O(h^\infty) = O(h^\infty),$$

a contradiction. Hence there are no quasimodes for $z_0 \in \partial\Lambda(P,\Omega)$.

Thus, we have proved

Lemma 10. *Suppose* $\operatorname{Re} z_0 = |X|^{-2}(\operatorname{Im} z_0)^2$. *Then there are no quasimodes of* (P,Ω) *for* z_0.

Remark: This argument can be adjusted slightly to give that if $d(z_0, \partial\Lambda(\Omega,P)) = O(h)$, then there are no quasimodes for $z_0 \in \partial\Lambda(P,\Omega)$.

7.2. **No Quasimodes Away from the Illuminated Boundary.**
We will need the following elementary lemma. We follow [6, Section 6.3.2]. Let $Q(x,hD) = -h^2\Delta + \langle a(x), h\partial\rangle + b(x)$.

Lemma 11. *Suppose* $u \in C^\infty(\Omega) \cap H_0^1(\Omega)$ *and* $\partial\Omega \in C^2$. *Then, we have*

$$\|u\|_{H_h^2(\Omega)} \leq C(\|Qu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Proof. Using a partition of unity and change of coordinates, we assume that $\Omega = B(0,1) \cap \mathbb{R}_+^d$ without loss of generality. Then, let $\chi \in C^\infty(\Omega)$ with $\chi \equiv 1$ on $V := \{|x| < \frac{1}{2}\}$ and $\chi \equiv 0$ on $|x| > \frac{3}{4}$. Next, let $v = -h^2\partial_k\chi^2\partial_k\bar{u}$ for $k = 1, ..., n-1$. Then, $v \in H_h^1$ with $v|_{\partial\Omega} = 0$ and hence

$$\int \langle h\partial u, h\partial v\rangle = \int Quv - \langle a(x), h\partial u\rangle v - b(x)uv.$$

Now,

$$\int \langle h\partial u, h\partial v\rangle \;=\; \int -h^4 \langle \partial u, \partial\partial_k(\chi^2 \partial_k \bar{u})\rangle = \int h^4 \langle \partial_k \partial u, 2\chi\partial_k\bar{u}\partial\chi + \chi^2 \partial\partial_k \bar{u}\rangle$$

$$\geq \;\; \int \chi^2(1 - C\epsilon)|h^2 \partial\partial_k u|^2 - h^2\epsilon^{-1}|h\partial_k u|^2 \geq \int \frac{1}{2}\chi^2|h^2\partial\partial_k u|^2 - Ch^2|h\partial_k u|^2.$$

Then,

$$\left| \int Quv - \langle a(x), h\partial u\rangle v - b(x)uv \right| \;\leq\; \int (|Qu| + |\langle a(x), h\partial u\rangle| + |b(x)u|)|v|$$

$$=\;\; \int (|Qu| + |\langle a(x), h\partial u\rangle| + |b(x)u|) \left| h^2 2\chi\partial_k\chi\partial_k\bar{u} + h^2\chi^2\partial_k^2\bar{u} \right|$$

$$\leq\;\; \int C|Qu|^2 + C|h\partial u|^2 + C|u|^2 + Ch^2|hDu|^2 + C\epsilon|h^2\chi^2\partial_k^2 u|^2$$

Thus,

$$\|h^2\partial_k\partial u\|_{L^2(V)} \leq C(\|Qu\|_{L^2(\Omega)} + \|u\|_{H_h^1(\Omega)})$$

for $k = 1, ..., n - 1$. Now, for $k = n$, we note that

$$h^2\partial_n^2 u = -Qu - \sum_{i=1}^{n-1} h^2\partial_i^2 u + \langle a(x), h\partial u\rangle + b(x)u.$$

Thus,

$$|h^2\partial_n^2 u| \leq C\left(\sum_{i=1}^{n-1} |h^2\partial_i^2 u| + |hDu| + |u| + |Qu| \right)$$

and we have

$$\|u\|_{H_h^2(V)} \leq C(\|Qu\|_{L^2(\Omega)} + \|u\|_{H_h^1(\Omega)})$$

and the result follows from [24, Theorem 7.1] and its proof. $\qquad\square$

We apply the above lemma to obtain the following,

Lemma 12. *Suppose that u has $u|_{\partial\Omega} = 0$, $\|u\|_{L^2} = 1$, and $ES_h(u) \cup ES_h(Pu) \subset A$ and $Pu = O_{L^2}(1)$. Then, for any U with $A \Subset U$, and $\chi \in C^\infty$ with $\chi \equiv 1$ on U,*

$$\|(1 - \chi)u\|_{H_h^2} = O(h^\infty).$$

In particular, if u is a quasimode for (1.2) with $ES_h(u) \subset A$, then, for any U with $A \Subset U$, there is a quasimode u_1 with $suppp\, u_1 \subset U$.

Proof. Let $A \Subset U_0 \Subset U \Subset B$. Let $\chi \in C^\infty$ have $\chi \equiv 1$ on U and supp $\chi \subset B$. Let $\chi_0 = \chi$ and for $i = 1, ...$ let $\chi_i \in C^\infty$ have supp $\chi \subset B \setminus U_0$ and have $\chi_i \equiv 1$ on supp $\partial\chi_{i-1}$. Then, by Lemma 11

$$\|(1 - \chi)u\|_{H_h^2} \leq C(\|P(1 - \chi)u\|_{L^2} + \|(1 - \chi)u\|_{L^2}) \leq C\|(1 - \chi)Pu\|_{L^2} + C\|[P, \chi]u\|_{L^2} + O(h^\infty)$$

$$= O(h^\infty) + \|[P, \chi]u\|_{L^2} \leq O(h^\infty) + Ch\|\chi_1 u\|_{H_h^1}.$$

But, using the same argument again, we have that since $\chi_n \equiv 0$ on U_0 for all n,

$$\|\chi_{n-1}u\|_{H_h^1} \leq \|(1 - (1 - \chi_{n-1}))u\|_{H_h^2} \leq O(h^\infty) + Ch\|\chi_n u\|_{H_h^1}$$

Hence, by induction, for all $N > 0$,

$$\|(1 - \chi)u\|_{H_h^2} \leq O(h^\infty) + C_N h^N \|\chi_N u\|_{H_h^1}.$$

But, by Lemma 11, since $Pu = O_{L^2}(1)$, $u = O_{H_h^2}(1)$ and hence

$$\|(1 - \chi)u\|_{H_h^2} \leq O(h^\infty) + C_N h^N \|u\|_{H_h^1} = O(h^\infty)$$

as desired.

To prove the second claim observe that if u is a quasimode,

$$\|P\chi u\|_{L^2} \leq \|P(1 - \chi)u\|_{L^2} + \|Pu\|_{L^2} = O(h^\infty)$$

since $(1 - \chi)u = O_{H_h^2}(h^\infty)$. $\square$

We now apply the above lemma to restrict the essential support of quasimodes.

Lemma 13. *If u is a quasimode for* (1.2) *then* $ES_h(u) \cap \overline{\partial\Omega_+} \neq \emptyset$.

Proof. Suppose that u is a quasimode for (1.2) and u concentrates away from $\overline{\partial\Omega_+}$. Then, by Lemma 12, we may assume that u is supported away from $\overline{\partial\Omega_+}$.

Now, applying Lemma 9 with $\epsilon = h \log h^{-1}$ and $\psi = \langle X, x \rangle^2$, we have

$$- h^3 \left(\langle 2\epsilon\partial\psi + X, \nu \rangle e^{\frac{\epsilon\psi}{h}} \partial\nu u, e^{\frac{\epsilon\psi}{h}} \partial\nu u \right)_{\partial\Omega_-} + Ch\epsilon \| e^{\frac{\epsilon\psi}{h}} u \|_{L^2}^2$$
$$\leq O(h^\infty) + h^3 \left(\langle 2\epsilon\partial\psi + X, \nu \rangle e^{\frac{\epsilon\psi}{h}} \partial\nu u, e^{\frac{\epsilon\psi}{h}} \partial\nu u \right)_{\partial\Omega_0}.$$

But, since $\partial\psi = 2\langle X, x \rangle X$, the term on $\partial\Omega_0$ vanishes and, hence, we have

$$Ch\epsilon \| e^{\frac{\epsilon\psi}{h}} u \|_{L^2}^2 \leq O(h^\infty).$$

Hence, $u = O(h^\infty)$ and there are no quasimodes concentrating away from $\partial\Omega_+$ - i.e.

$$ES_h(u) \cap \overline{\partial\Omega_+} \neq \emptyset.$$

$\square$

7.3. **Characterization of the Essential Support of Quasimodes.** In order to use Lemma 9 to restrict where quasimodes can concentrate, we would like to design a set A with $\Gamma_+ \subset A$ and a weight function ψ such that for any $U \subset \overline{\Omega}$ separated from A, there exists $\epsilon > 0$ such that $\sup_A \psi < \inf_U \psi - \epsilon$. Since ψ must be convex in $\overline{\Omega}$ to apply Lemma 9, any set A with this property must be relatively convex inside $\overline{\Omega}$.

7.3.1. *Preliminaries Relatively on Convex Sets.* Let B be a bounded set and A be convex relative to B. We wish to determine whether there is a smooth strictly convex function (inside B) with ∂A as a level set.

Lemma 14. *Let A be a closed and relatively convex set inside B, a bounded set. Then there is a function g_A that is convex inside B and has $g_A|_A \equiv 0$, $g_A(x) > 0$ for $x \notin A$.*

Proof. First, define the epigraph of a function f as follows.

$$\text{epi}(f) = \{(x, \mu) \in B \times \mathbb{R} : \mu \geq f(x)\}.$$

We first show that a function f is locally convex in B if and only if its epigraph is relatively convex in $B \times \mathbb{R}$.

Suppose that f is convex in B. Then, for every $x, y \in B$ with $L_{x,y} \subset B$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. (Here $L_{x,y}$ is as in Definition 3) Therefore, if $(x, \mu), (y, \nu) \in \text{epi}(f)$, then $(tx + (1-t)y, t\mu + (1-t)\nu) \in \text{epi}(f)$.

Now, suppose that $\text{epi}(f)$ is relatively convex in $B \times \mathbb{R}$. Then, suppose that $x, y \in B$ with $L_{x,y} \subset B$. Then, let $f(x) = \mu$ and $f(y) = \nu$. Then, $t(x, \mu) + (1-t)(y, \nu) \in \text{epi}(f)$. Hence,

$$f(tx + (1-t)y) \leq t\mu + (1-t)\nu = tf(x) + (1-t)f(y)$$

and f is convex in B.

Now, we determine the epigraph of the function g_A. First, let

$$G = A \times [0, \infty) \cup B \setminus A \times [1, \infty).$$

Then, let $G_A = Coh_{B \times \mathbb{R}}(G)$. Observe that since A is relatively convex in B, $A \times [0, \infty)$ is relatively convex in $B \times \mathbb{R}$. Now, by Carathéodory's Theorem, any point in $Coh(G)$ can be written as the convex combination of at most $d + 2$ points in G. Since $A \times [0, \infty)$ is relatively convex in $B \times \mathbb{R}$, and $Coh_{B \times \mathbb{R}}(G) \subset Coh(G)$, any point in $G_A \setminus (A \times [0, \infty))$ must be a convex combination that involves points in $B \setminus A \times [1, \infty)$.

Suppose $x \notin A$ and $(x, \nu) \in G_A$. Then, $d(x, A) > 0$ since A is closed and

$$(x, \nu) = \sum_{i=1}^{d+2} t_i(x_i, \nu_i)$$

where, for some $r > 0$, $x_1, ..., x_r \notin A$. Hence, $\nu_1, ..., \nu_r = 1$. Relabel (x_i, ν_i) $i = 1, ..., r$ so that $t_1 = \max(t_1, ..., t_r)$. Then, since B is bounded there exists $r > 0$ such that $B \subset B(0, r)$ and hence $t_1 > d(x, A)/((d+2)r)$. Therefore $\nu \geq d(x, A)/((d+2)r) > 0$. Thus letting

$$g_A(x) = \inf\{y : (x, y) \in G_A\},$$

g_A is convex in B with $g_A > 0$ on $B \setminus A$ and $g_A = 0$ on A. $\qquad \square$

Corollary 2. *Let B and $B_1 \supseteq B$ be bounded sets. Let $A \subset B_1$ be closed and convex relative to B_1. Then there exists $\psi \in C^\infty$ strictly convex in B such that for all W with $A \Subset W$, $\sup_A \psi < \inf_{B \setminus W} \psi$.*

Proof. Let $d(B_1, B) = 2\delta$. Then, let $\varphi_\epsilon \in C_0^\infty$ be a nonnegative approximate identity family with support contained in $B(0, \delta)$ and define $f_A^\epsilon = g_A * \varphi_\epsilon$. Then, $f_A^\epsilon \to g_A$ uniformly on bounded sets. Also, f_A^ϵ is smooth. To see that f_A^ϵ is convex inside B, observe that for $x, y \in B$ with $L_{x,y} \subset B$,

$$f_A^\epsilon(tx + (1-t)y) = \int \varphi_\epsilon(z)g_A(tx + (1-t)y - z)dz$$

$$\leq \int \varphi_\epsilon(z)(tg_A(x - z) + (1-t)g_A(y - z))dz = tf_A^\epsilon(x) + (1-t)f_A^\epsilon(y)$$

by the convexity of g_A inside B_1 and the nonnegativity of φ_ϵ. Finally, to make a strictly convex approximation of g_A, define $g_A^\epsilon := f_A^\epsilon + \epsilon|x|^2$. Then, $g_A^\epsilon \to g_A$ uniformly on bounded sets, and $g_A^\epsilon \in C^\infty$ with g_A^ϵ strictly convex inside B. $\qquad\square$

Remark: Although we have not constructed a smooth strictly convex function with level set ∂A, we have one that has a level set which is uniformly arbitrarily close.

We also need a few more properties of relatively convex sets

Lemma 15. *Suppose that $A \subset B$ is relatively convex in B, B open and bounded. Then, $\overline{A}$ is relatively convex in B.*

Proof. Let $x, y \in \overline{A}$ such that $L_{x,y} \subset B$. Then, there are sequences $x_n \to x$ and $y_n \to y$ with $x_n, y_n \subset A$. We need to show that $L_{x,y} \subset \overline{A}$. For $0 \leq \lambda \leq 1$, We have that

$$|\lambda(x_n - x) + (1 - \lambda)(y_n - y)| \leq |x_n - x| + |y_n - y|.$$

But, since $L_{x,y}$ is compact and B is open, there is $\epsilon > 0$ such that

$$\{z : d(z, L_{x,y}) < \epsilon\} \subset B$$

and hence, we have that for n large enough $L_{x_n,y_n} \subset B$. But, since A is convex, this implies $L_{x_n,y_n} \subset A$ and hence for $0 \leq \lambda \leq 1$

$$\lim_{n \to \infty} \lambda x_n + (1 - \lambda)y_n = \lambda x + (1 - \lambda)y \in \overline{A}.$$

$\qquad\square$

Lemma 16. *We have that*

$$\bigcap_{B \Subset B_1} Coh_{B_1}(A) = Coh_{\overline{B}}(A).$$

Proof. Let

$$\mathcal{C} = \{C : A \subset C \,, x, y \in C \,, L_{x,y} \Subset B_1 \text{ for all } B_1 \ni B \text{ implies } L_{x,y} \subset C\}.$$

Then,

$$\bigcap_{B \Subset B_1} Coh_{B_1}(A) = \bigcap_{C \in \mathcal{C}} C.$$

But, if $L_{x,y} \not\subseteq \overline{B}$, then $L_{x,y} \not\subseteq B_1$ for some $B_1 \ni B$. Hence,

$$\mathcal{C} = \{C : A \subset C \,, x, y \in C \,, L_{x,y} \Subset \overline{B} \text{ implies } L_{x,y} \subset C\}$$

and the result follows.

$\qquad\square$

7.3.2. *Application to Quasimodes.* We now apply the above results on convex sets to quasimodes.

Lemma 17. *If u has $u|_{\partial\Omega} = 0$, $ES_h(u) \subset A$, $Pu = O_{L^2}(1)$, and $ES_h(Pu) \subset B$, then, for all $A_1 \Supset A$ and $B_1 \Supset B$,*
$$ES_h(u) \subset \overline{Coh_{\overline{\Omega}}((A_1 \cap \Gamma_+) \cup B_1)}.$$
In particular, if u is a quasimode for (1.2)*, then for all $A_1 \Supset A$*
$$ES_h(u) \subset \overline{Coh_{\overline{\Omega}}(A_1 \cap \Gamma_+)}.$$

Proof. Choose Ω_1 open with $\overline{\Omega} \Subset \Omega_1$, A_1 closed with $A \Subset A_1$, and B_1 closed with $B \Subset B_1$. Let $F = \overline{Coh_{\Omega_1}((A_1 \cap \Gamma_+) \cup B_1)}$. Then, by Lemma 15, F is closed and convex. Let $U \subset \overline{\Omega}$ such that $d(U, F) > 0$. By Corollary 2, there exists $\psi \in C^\infty$ strictly convex in $\overline{\Omega}$ such that for some $\delta > 0$, $\sup_F \psi < \inf_U \psi - \delta$. We have, by Lemma 9 that, for ϵ_0 small enough independent of h, and h small enough, for $\epsilon(h) < \epsilon_0$,

$$Ch\epsilon\|e^{\frac{\epsilon\psi}{h}}u\|^2_{H^1_h} \leq C\|e^{\frac{\epsilon\psi}{h}}Pu\|^2 + Ch^3\left(\langle 2\epsilon\partial\psi + X, \nu\rangle e^{\frac{\epsilon\psi}{h}}\partial\nu u, e^{\frac{\epsilon\psi}{h}}\partial\nu u\right)_{\Gamma_+}.$$

Now, suppose that u has $ES_h(u) \subset A$, $ES_h(Pu) \subset B$, and let $\epsilon = \gamma h \log h^{-1}$. Then, by Lemma 12, up to $O_{H^2_h}(h^\infty)$ we may assume that supp $u \subset A_1 \cup B_1$. Thus,

$$h^3\left(\langle 2\epsilon\partial\psi + X, \nu\rangle e^{\frac{\epsilon\psi}{h}}\partial\nu u, e^{\frac{\epsilon\psi}{h}}\partial\nu u\right)_{\Gamma_+} \leq O(h^\infty) + h^3\left(\langle 2\epsilon\partial\psi + X, \nu\rangle e^{\frac{\epsilon\psi}{h}}\partial\nu u, e^{\frac{\epsilon\psi}{h}}\partial\nu u\right)_{(A_1 \cup B_1) \cap \Gamma_+}.$$

Hence, we have

$$Ch\epsilon\|e^{\frac{\epsilon\psi}{h}}u\|^2_{L^2} \leq O(h^\infty) + \|e^{\epsilon\psi/h}Pu\|_{B_1} + h^3\left(\langle 2\epsilon\partial\psi + X, \nu\rangle e^{\frac{\epsilon\psi}{h}}\partial\nu u, e^{\frac{\epsilon\psi}{h}}\partial\nu u\right)_{(A_1 \cup B_1) \cap \Gamma_+}.$$

But, by Lemma 11, $\|u\|_{H^2_h} = O(1)$. Hence,

$$h^3(\partial_\nu u, \partial_\nu u)_{\partial\Omega} \leq Ch^3\|u\|^2_{H^{3/2}} = C\|u\|^2_{H^{3/2}_h} \leq C\|u\|^2_{H^2_h} = O(1).$$

Thus, we have that

$$Ch\epsilon\inf_U e^{\frac{2\epsilon\psi}{h}}\|u\|^2_{L^2(U)} \leq Ch\epsilon\|e^{\frac{\epsilon\psi}{h}}u\|^2_{L^2(U)} \leq Ch\epsilon\|e^{\frac{\epsilon\psi}{h}}u\|^2_{L^2} \leq O(h^\infty) + C\sup_{(A_1 \cap \Gamma_+) \cup B_1} e^{\frac{2\epsilon\psi}{h}}.$$

But, $\inf_U \psi \geq \delta + \sup_B \psi$ and we have

$$C_\gamma h^{2-2\gamma\delta}\log h^{-1}\|u\|_{L^2(U)} = Ch\epsilon e^{\frac{2\epsilon\delta}{h}}\|u\|^2_{L^2(U)} \leq O(h^\infty) + C.$$

Hence, letting $\gamma \to \infty$, we have that $\|u\|_{L^2(U)} = O(h^\infty)$ as desired.

Thus, quasimodes cannot have essential support away from F. That is for any $A_1 \Supset A$ and $B_1 \Supset B$,

$$ES_h(u) \subset \bigcap_{\Omega \Subset \Omega_1} \overline{Coh_{\Omega_1}((A_1 \cap \Gamma_+) \cup B_1)} = \overline{Coh_{\overline{\Omega}}((A_1 \cap \Gamma_+) \cup B_1)}$$

Here, equality of the two sets follows from Lemma 16. The second claim follows from the fact that a quasimode has $ES_h(Pu) = \emptyset$. $\qquad\qquad\square$

Remark: Observe that if $\Gamma_+ \subset A$, then the second part of Lemma 17 gives that for quasimodes
$$ES_h(u) \subset \overline{Coh_{\overline{\Omega}}(\Gamma_+)}.$$

7.4. **Characterization of the Interior Wavefront set of a Quasimode.** We wish to determine where a quasimode can concentrate. To do this we first need the following simple lemmas

Lemma 18. *For a solution to (1.2),*

$$\pi_x(WF_h(u)) \cap \Omega^o = ES_h(u) \cap \Omega^o.$$

Proof. Say $x_0 \in \pi_x(WF_h(u)) \cap \Omega^o$. Then it is clear that $x_0 \in \mathrm{ES}_h(u)$.

Now, suppose $x_0 \notin \pi_x(WF_h(u)) \cap \Omega^o$. Let $K > 0$ such that $|p(x,\xi)| \geq 2C\langle\xi\rangle^2$ for $|\xi| \geq K$ and $x \in \Omega^o$. Let U be a neighborhood of x_0 such that for all $\chi \in C_0^\infty(U \times \{|\xi| \leq 2K\})$,

$$\|\chi^w u\|_{L^2} = O(h^\infty).$$

Such a neighborhood, U exists by the compactness of $\{|\xi| \leq 2K\}$ and [24, Theorem 8.13].

Let $\varphi \in C_0^\infty(U)$ and $\psi \in C^\infty(\mathbb{R}^d)$ with supp $\psi \subset \{|\xi| \leq 2K\}$.

To complete the proof, we need only show that there exists $V \Subset U$ such that

$$\|(1 - \psi(\xi))^w u\|_{L^2(V)} = O(h^\infty).$$

To see this, let $\psi_1 \in C_0^\infty(\mathbb{R}^d)$ have $\psi \equiv 1$ on supp ψ_1 and supp $\psi_1 \subset \{|\xi| \leq 2K\}$, let $\varphi_1 \in C_0^\infty(U)$ with $\varphi_1 \equiv 1$ on supp φ, and finally let $\varphi_2 \in C_0^\infty(U)$ with $\varphi_2 \equiv 1$ on supp φ_1.

Then, observe that

$$(1 - \psi_1)^2 |p|^2 \varphi_2^2 \varphi_1^2 \geq \gamma \langle\xi\rangle^4$$

on supp $(1 - \psi)\varphi$. Hence, by the Sharp Garding inequality,

$$\|\varphi_2 P\langle hD\rangle^{-2}\langle hD\rangle^2 (1 - \psi)^w \varphi u\|_{L^2}^2 \geq \gamma^2 \|(1 - \psi)^w \varphi u\|_{L^2}^2 - Ch\|\langle hD\rangle^2 (1 - \psi)^w \varphi u\|_{L^2}^2.$$

But, by Lemma 11

$$\|(1 - \psi)^w u\|_{H_h^2} \leq C(\|(1 - \psi)^w \varphi u\|_{L^2} + \|P(1 - \psi)^w \varphi u\|_{L^2})$$

and we have that

$$\frac{\gamma}{2}\|(1 - \psi)^w \varphi u\|_{L^2} \leq C\|\varphi_2 P(1 - \psi)^w \varphi u\|_{L^2}$$

But,

$$\varphi_2 P(1 - \psi)^w \varphi u = \varphi_2 Pu + \varphi_2 P((1 - \psi^w)\varphi - 1)u = O_{L^2}(h^\infty).$$

since $Pu = O(h^\infty)$ and $\varphi_2 P(1 - (1 - \psi^w)\varphi) = c^w$ with supp $c \subset U \times \{|\xi| \leq 2K\}$. $\square$

Lemma 19. *If $\partial\Omega$ is smooth, then for any plane A with X tangent to A and $A \cap \Omega^o \neq \emptyset$, we have*

$$(\Omega^o \setminus Coh(\Gamma_+)) \cap A \neq \emptyset.$$

Proof. For simplicty, we again assume $X = e_1$. First, observe that Γ_+ is compact. Choose $x \in \Gamma_+ \cap A$ such that $\pi_1(x) \leq \pi_1(y)$ for all $y \in \Gamma_+ \cap A$. Then,

$$\pi_1(Coh(\Gamma_+) \cap A) \subset [\pi_1(x), \infty).$$

We show that there is a $z \in \Omega^o \cap A$ with $\pi_1(z') < \pi_1(x)$ and hence that $z' \notin Coh(\Gamma_+) \cap A$.

Suppose $x \in \partial\Omega_+$. Then, $\langle e_1, \nu(x)\rangle > 0$ and hence there is $z \in \Omega^o \cap A$ with $\pi_1(z') < \pi_1(x)$. Now, suppose that $x \in \partial\Omega_0$. Then, e_1 is tangent to $\partial\Omega \cap A$ at x. Hence, there is a $z \in \partial\Omega \cap A$ with $\pi_1(z) < \pi_1(x)$. But, this implies that there is a $z' \in \Omega^0 \cap A$ with $\pi_1(z') < \pi_1(x)$. $\square$

We now finish the proof of part (2) of Theorem 1.

Proof. Let u be a quasimode. Observe that if $(p - z)(x_0, \xi_0) = 0$ and $\xi_0 \neq (\operatorname{Im} z, 0, ..., 0)$, then $\operatorname{Re} z > (\operatorname{Im} z)^2$. Hence, by Lemma 8, and Corollary 1, if

$$x_0 \in \Omega^o, \quad (x_0, \xi_0) \in WF_h(u), \quad \xi \neq (\operatorname{Im} z, 0, 0, ..., 0),$$

then, there exists a plane A tangent to e_1 with $x_0 \in A$ such that

$$\pi_x(WF_h(u)) \supset A \cap \Omega^o.$$

But, $ES_h(u)$ is closed and $ES_h(u) \cap \Omega^o = \pi_x(WF_h(u) \cap (\Omega^o \times \mathbb{R}^d))$, hence

$$ES_h(u) \supset A \cap \overline{\Omega}.$$

Together with Lemma 17, this gives for $\Gamma_+ \Subset A_1$

$$A \cap \overline{\Omega} \subset ES_h(u) \subset \overline{Coh_{\overline{\Omega}}(A_1)}.$$

But, notice that

$$\overline{Coh_{\overline{\Omega}}(A_1)} \subset \overline{Coh(A_1)}.$$

Hence, since $A_1 \ni \Gamma_+$ was arbitrary,

$$ES_h(u) \subset \bigcap_{A_1 \ni \Gamma_+} \overline{Coh(A_1)} = Coh(\Gamma_+)$$

since Γ_+ is compact. Therefore we have a contradiction of (19). Putting this together with Lemma 8, we have

$$WF_h(u) \cap (\Omega^o \times \mathbb{R}^d) \subset (\Omega^o \times \{(\operatorname{Im} z, 0, 0, ..., 0)\}) \cap p^{-1}(0).$$

Now, note that if $\xi = (\operatorname{Im} z, 0, ..., 0)$, and $p(x, \xi) - z = 0$, then $\operatorname{Re} z = (\operatorname{Im} z)^2$ and hence $z \in \overline{\Lambda(P, \Omega)}$. Thus, except for $z \in \partial\overline{\Lambda(P, \Omega)}$,

$$WF_h(u) \cap (\Omega^o \times \mathbb{R}^d) = \emptyset.$$

But, we have shown in Lemma 10 that there are no quasimodes for $z \in \partial\overline{\Lambda(P, \Omega)}$. Hence we have proved that quasimodes cannot have wave front set in the interior of Ω.

So, using Lemma 18, we have that

(7.1) $$ES_h(u) \cap \Omega^o = \pi_x(WF_h(u) \cap (\Omega^o \times \mathbb{R}^d)) = \emptyset.$$

Thus, $ES_h(u) \subset \partial\Omega$.

To finish the proof of Theorem 1, we apply Lemma 17 with $ES_h(u) \subset \partial\Omega$ to obtain

(7.2) $$ES_h(u) \subset \overline{Coh_{\overline{\Omega}}(\Gamma_+)}.$$

Putting (7.1) and (7.2) together, we have that quasimodes cannot concentrate away from the intersection of the $\overline{\Omega}$ convex hull of the glancing and illuminated boundary with the boundary - i.e.

$$ES_h(u) \subset \partial\Omega \cap \overline{Coh_{\overline{\Omega}}(\Gamma_+)}$$

as desired. $\qquad\square$

7.5. **Further Localization.** We now apply Lemma 9 locally to restrict the essential support of quasimodes further, proving parts (4) and (5) of Theorem 1.

We will need the following lemma.

Lemma 20. *Let $U \subset \overline{\Omega}$, then for any quasimode u of (1.2), any $V \Subset U$, and any $W \ni \partial\Omega$, we have*

$$ES_h(u) \cap V \subset \overline{Coh_{\overline{U}}((\overline{U} \cap \Gamma_+) \cup ((U \setminus V) \cap W))} \cap \partial\Omega.$$

Proof. Let u be a quasimode for (1.2), $\chi \in C^\infty(\overline{\Omega})$. Now, let W have supp $\partial\chi \Subset W$ and let U_1 be a neighborhood of $W \cap \partial\Omega$. Then, let $\chi_1 \in C^\infty(\overline{\Omega})$ with $\chi_1 \equiv 1$ on U_1. Then,

$$\|(1 - \chi_1)P\chi u\| = O(h^\infty) + \|(1 - \chi_1)[P, \chi]u\|.$$

Now, by Lemma 12, and the fact that $ES_h(u) \subset \partial\Omega$,

$$\|(1 - \chi_1)[P, \chi]u\|_{L^2} \le C\|(1 - \chi_1)u\|_{H^1_h(W)} = O(h^\infty).$$

Hence, since U_1 was an arbitrary neighborhood of supp $\partial\chi \cap \partial\Omega$,

$$ES_h(Pu) \subset \text{supp } \partial\chi \cap \partial\Omega.$$

Then, observe that χu is a function on $\Omega_1 = \text{supp } \chi \cap \overline{\Omega}$ with

$$\chi u|_{\partial\Omega_1} = 0, \quad ES_h(\chi u) \subset \partial\Omega \cap \text{ supp } \chi, \quad ES_h(Pu) \subset \text{ supp } \partial\chi \cap \partial\Omega.$$

Hence, applying Lemma 17 to χu on Ω_1, and using the fact that $ES_h(u) \subset \partial\Omega$, we have for every $B \ni \text{ supp } \partial\chi \cap \partial\Omega$ and every $A \ni \text{ supp } \chi \cap \partial\Omega$,

$$(7.3) \qquad\qquad ES_h(\chi u) \subset \overline{Coh_{\text{supp } \chi \cap \overline{\Omega}}((\Gamma_+ \cap A) \cup B)} \cap \partial\Omega.$$

Now, let $V \Subset V_1 \Subset V_2 \Subset U \subset \overline{\Omega}$, and $W \ni \partial\Omega$. Let $\chi \in C^\infty(\overline{\Omega})$ with $\chi \equiv 1$ on V_1 and supp $\chi \subset V_2$. Then, applying (7.3) we have that

$$ES_h(u) \cap V \subset \overline{Coh_{\overline{U}}((U \cap \Gamma_+) \cup ((U \setminus V) \cap W))} \cap \partial\Omega.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We now use Lemma 20 to finish the proof of Theorem 1.

Proof. For simplicity, assume $X = e_1$. To prove the first part of the proposition, suppose that $\partial\Omega$ is either strictly concave or strictly convex at x_0. Then there exists $U \subset \overline{\Omega}$, $x_0 \in U$ such that $U \cap \partial\Omega \subset \partial\Omega_-$ and for $x \in \partial U \cap \partial\Omega$, $|\pi_1(x) - \pi_1(x_0)| > \delta$ for some $\delta > 0$. Hence, there is a $V \Subset U$ such that for some $W \ni \partial\Omega$, $x \in (U \setminus V) \cap W$ implies $|\pi_1(x) - \pi_1(x_0)| > \delta/2$ and thus

$$x_0 \notin \overline{Coh_{\overline{U}}((U \cap \Gamma_+) \cup ((U \setminus V) \cap W))} \cap \partial\Omega.$$

Hence $x_0 \notin ES_h(u)$.

Now, suppose $\Omega \subset \mathbb{R}^2$. Then choose $x_0 \in \partial\Omega_-$ and let $\gamma : [-1, 1] \to \partial\Omega$ be a curve defining $\partial\Omega$ with $\gamma(0) = x_0$, $\gamma(-1), \gamma(1) \in \partial\Omega_0$, and $\gamma((-1, 1)) \subset \partial\Omega_-$. Defining

$$t_\pm := \inf\{t : \gamma'(\pm t) \ne \gamma'(0)\},$$

we have $|t_\pm| < 1$ since if not, then $\langle X, \gamma'(\pm 1)\rangle \ne 0$. Then, there exists $\epsilon > 0$, such that for

$$I_- \cup I_+ := [t_- - \epsilon, t_- - \epsilon/2] \cup [t_+ + \epsilon/2, t_+ + \epsilon].$$

there is a $W \supseteq \gamma(I_+ \cup I_-)$ with $W \cap \Gamma_+ = \emptyset$ and a $\delta > 0$ such that

$$\inf_{z \in W} \sup_{s \in [0,1]} d(sz + (1-s)x_0, \partial\Omega) > \delta.$$

Let $V \Subset U \subset \overline{\Omega}$ such that $x_0 \in U$, $(U \setminus V) \cap \partial\Omega \subset \gamma(I_- \cup I_+)$, and

$$U \subset \{x \in \overline{\Omega} : d(x, \partial\Omega) < \delta/2\}.$$

Then, letting u be a quasimode for (1.2) and applying the Lemma 20, we have

$$ES_h(u) \cap V \subset \overline{Coh_{\overline{U}}((U \setminus V) \cap W)}$$

and hence $x_0 \notin ES_h(u)$. But, $x_0 \in \partial\Omega_-$ was arbitrary. Therefore, $ES_h(u) \subset \Gamma_+$ as desired. $\qquad\square$

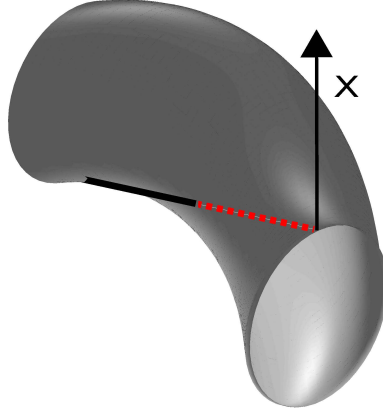**Remark:** Figure 4 shows an example of why we cannot make a similar argument in dimesions larger than 2.

FIGURE 4. The figure shows a piece of a domain $\Omega \subset \mathbb{R}^3$. A portion of $\partial\Omega_-$ is shown in the black line and a portion of $\partial\Omega_+$ is shown in the dashed red line. Notice that for any point in the portion of $\partial\Omega_-$ shown, $\partial\Omega_+$ can be reached along a straight line lying entirely inside the boundary. This example shows that in demensions higher than 2 we cannot hope to make an argument similar to that used to prove the last part of Theorem 1.

8. Instability in an Evolution Problem

Our approach to obtaining blow-up of (1.4) will follow that used by Sandstede and Scheel in [20] and that by the author in [8]. We first demonstrate that, from small initial data, we obtain a solution that is ≥ 1 on a translated ball in time $t_1 = O(1)$. We then use the fact that the solution is ≥ 1 on this region to demonstrate that, after an additional $t_2 = O(h)$, the solution to the equation blows up.

First, we prove that there exists initial data so that the solution to (1.4) is ≥ 1 in time $O(1)$. Let $\varphi_t := \exp(-tX)$ denotes the flow of $i\langle X, D\rangle$. Note that for the purposes of Theorem 2, we do not need to assume that X is constant.

Lemma 21. *Fix $\mu > 0$, $\alpha < \mu$, $0 < \epsilon \leq \frac{1}{2}(\mu - \alpha)$, and $(x_0, a, \delta) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+$ such that both $\varphi_t(B(x_0, 2a)) \subset \Omega$ for $t \leq 2\delta$ and φ_t is defined on $B(x_0, 2a)$ for $0 \leq t < 2\delta$. Then, for each*

$$0 < h < h_0$$

where h_0 is small enough, there exists

$$u_0(x) \geq 0, \quad \|u_0\|_{C^P} \leq \exp\left(-\frac{1}{Ch}\right), \quad p = 0, 1, \dots$$

and $0 < t_1 < \delta$ so that the solution to (1.4) with initial data u_0 satisfies $u(x, t_1) \geq 1$ on $x \in \varphi_{t_1}(B(x_0, a))$.

Proof. The proof for this lemma follows that in [8, Lemma 3] except we no longer need to control the size of the potential. Instead, we show that the ansatz stays 0 on $\partial\Omega$.

Let v solve

$$(8.1) \qquad\qquad (h\partial_t + P(x, hD) - \mu)v = 0, \quad v(x, 0) = v_0, \quad v|_{\partial\Omega} = 0.$$

Let $w_0 : \mathbb{R}^d \to \mathbb{R}$ and define $O := \{x : w_0 > 0\}$. We make the following assumptions on w_0,

$$(8.2) \qquad\qquad w_0 \geq 0, \quad \|w_0\|_{C^p} \leq \exp(-\frac{1}{Ch}), \quad w_0 \in C(\mathbb{R}^d)$$

$$(8.3) \qquad w_0 \in C^\infty(\overline{O}), \quad \operatorname{supp} w_0 \subset B(x_0, 2a), \quad w_0 > \exp\left(-\frac{\delta}{2h}\right) \text{ on } B(x_0, a),$$

$$(8.4) \qquad \partial O \text{ is smooth}, \quad -\Delta w_0(x) \leq Cw_0(x) - \beta \text{ for } x \in O \text{ and } 0 < h < h_0.$$

where $C^\infty(\overline{O})$ are smoothly extendible functions on O. We refer the reader to [8, Lemma 3] for the construction of such a function.

Define $w : [0, 2\delta) \times \mathbb{R}^d \to \mathbb{R}$ by

$$w := \begin{cases} \exp\left(\frac{\alpha}{h}t\right) w_0(\varphi_t(x)) & \text{where } \varphi_t \text{ is defined,} \\ 0 & \text{else.} \end{cases}$$

Since $\operatorname{supp} w \subset B(0, 2a)$ and φ_t is defined on $B(0, 2a) \times [0, 2\delta)$, w is continuous. We proceed by showing that w is a viscosity subsolution of (8.1) in the sense of Crandall, Ishi, and Lions [1].

First, we show that w is a subsolution on $O_t := \varphi_t(O)$ for $t < \delta$.

$$\begin{aligned} hw_t + P(x, hD)w - \mu w &= hw_t - h^2\Delta w + ih\langle X, D\rangle w - \mu w \\ &= (\alpha - \mu)w - h^2\Delta w \\ &\leq \exp\left(\frac{\alpha}{h}t\right)((\alpha - \mu)w_0) - h^2\Delta w \end{aligned}$$

Now, by Taylor's formula, $\varphi_t(x) = x + O(t)$. Hence $-\Delta[w_0(\varphi_t(x))] = -\Delta w_0(x) + O(t)$. We have $t < \delta$, and $-\Delta w_0 \leq Cw_0 - \beta$ on O. Therefore, for δ small enough, $-\Delta w \leq Cw_0$. Hence, for h small enough independent of $0 < \delta < \delta_0$,

$$hw_t - P(x, hD)w \leq \exp\left(\frac{\alpha}{h}t\right)\left(\alpha - \mu + Ch^2\right)w_0 \leq 0$$

Now, since for $t < \delta$, supp $w \subset \Omega$ we have that w is a subsolution on O_t for $t < \delta$ and h small enough. Next, observe that on $(\mathbb{R}^d \setminus \overline{O_t})$, $w \equiv 0$ and hence is a subsolution of (8.1) on this set as well.

Finally, we need to show that w is a subsolution on $\partial O_t := \varphi_t(\partial O)$. We refer the reader to the proof of [8, Lemma 3] for this. Lastly, observe that since $\varphi_t(B(x_0, 2a)) \subset \Omega$ for $t < 2\delta$, we have that for $t < \delta$, $w|_{\partial\Omega} = 0$. Together with the previous arguments, this shows that w is a viscosity subsolution for (8.1) on $t < \delta$.

Now, by an adaptation of the maximum principle found in [1, Section 3] to parabolic equations, any solution, v to (8.1) with initial data $v_0 > w_0$ has $v \geq w$ for $t < \delta$. But, since $v \geq 0$, $v^3 \geq 0$ and hence the solution u to (1.4) with initial data v_0 has $u \geq v \geq w$ for $t < \delta$. Then, since for $t > \frac{\delta}{2}$, $w(x, t) \geq 1$ on $\varphi_t(B(x_0, a))$, we have the result. $\qquad\square$

Remark: To obtain a growing subsolution it was critical that $\mu > 0$. This corresponds precisely with the movement of the pseudospectrum of $(-(P - \mu), \Omega)$ into the right half plane.

Now, we demonstrate finite time blow-up using the fact that in time $O(1)$ the solution to (1.4) is ≥ 1 on an open region. Again, the proof of Theorem 2 follows that in [8, Theorem 1] except we replace the need to control the size of the potential with the requirement that the solution be 0 on $\partial\Omega$.

Proof. Let $u_0(x)$ and t_1 be the initial data and time found in Lemma 21 with (a, x_0, δ) such that φ_t is defined on $B(x_0, a)$, $\varphi_t(B(x_0, a)) \subset \Omega$ for $t \in [0, \delta]$, and $t_1 < \delta$. Then, $u(x, t_1) \geq 1$ on $\varphi_t(B(x_0, a))$.

Now, let $\Phi \in C_0^\infty(\mathbb{R})$ be a smooth bump function with $\Phi(y) = 1$ on $|y| \leq 1$, $0 \leq \Phi \leq 1$, supp $\Phi \subset (-2, 2)$, and $\Phi'' \leq C\Phi^{1/3}$ (one such function is given by $e^{-1/x}$ for $\epsilon > x > 0$). Define $\chi : \mathbb{R}^d \to \mathbb{R}$ by $\chi(y) := \Phi\left(2a^{-1}|y|\right)$.

Next, let $y' = \varphi_t(x_0 + y)$ and let

$$v(y, t) := \chi(y)u(y', t).$$

Then, we have that

$$hv_t = h^2\Delta v + \mu v + v^3 - 2h^2\langle\nabla\chi, \nabla u\rangle - h^2\Delta\chi u + (\chi - \chi^3)u^3.$$

Finally, define the operations, $[f]$ and (f, g) by

$$[f] := \fint_{B(0,a)} f(y)dy \quad (f, g) := \fint_{B(0,a)} \langle f(y), g(y)\rangle dy.$$

Then, we have that

$$
\begin{aligned}
h[v]_t &= h^2[\Delta v] + \mu[v] + [v^3] - h^2\left(\Delta\chi, u\right) - 2h^2\left(\nabla\chi, \nabla u\right) + \left(\chi - \chi^3, u^3\right) \\
&\geq \mu[v] + [v^3] + h^2\left(\Delta\chi, u\right) + \left(\chi - \chi^3, u^3\right)
\end{aligned}
\tag{8.5}
$$

Here, (8.5) follows from integration by parts, and the fact that $\nabla\chi = 0$ at $|y| = a$.

We will later need that $[v^3] \geq [v]^3$. To see this use Hölder's inequality. We will also need an estimate on $(\Delta\chi, u)$. Following [8, Section 4], we obtain

$$
(\Delta\chi, u) \leq C\fint(1 + \chi u^3) \leq C' + C\fint\chi u^3
\tag{8.6}
$$

where C' and C do not depend on h.

Now, we have

$$
\begin{aligned}
h[v]_t &= \mu[v] + [v^3] + h^2\left(\Delta\chi, u\right) + \left(\chi - \chi^3, u^3\right) \\
&\geq \mu[v] + [v^3] - O(h^2) + \left((1 - O(h^2))\chi - \chi^3, u^3\right) \\
&\geq \mu[v] + [v^3] - O(h^2) - O(h^2)[v^3] \\
&\geq \mu[v] + (1 - O(h^2))[v]^3 - O(h^2)
\end{aligned}
\tag{8.7}
$$

Here, (8.7) follows from the fact that $\chi \leq 1$ and $[v^3] \geq [v]^3$. Note that these equations are satisfied for $t < \delta$ since $\varphi_t(B(x_0, a)) \subset \Omega$ for $t < \delta$.

We have that $[v](t_1) \geq 1/4$ and $\mu > 0$. Therefore there exists $\gamma > 0$ such that, for h small enough and $t_1 \leq t \leq t_1 + \gamma$,

$$
h[v]_t \geq \frac{\mu}{2}[v] + \frac{1}{2}[v]^3.
$$

But, the solution to this equation with initial data $[v](0) \geq 1/4$ blows up in time $t_2 = O(h)$. Hence, so long as $t_1 + t_2 < \min(\delta, t_1 + \gamma)$ and h is small enough, $[v]$ blows up in time $t_1 + t_2$. Observe that since $t_1 < \delta$, $0 \leq t_1 + t_2 = t_1 + O(h) < \min(\delta, t_1 + \gamma)$ for h small enough. Thus, the solution to (1.4) blows up in time δ. $\square$

9. Application to Hitting Times for Diffusion Processes

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^∞ boundary. Then, define the stochastic process

$$
dX_t = b(X_t) + \sqrt{2h}dB_t
\tag{9.1}
$$

where B_t is Brownian motion. (Figure 5 shows an example path for X_t.) Let $Y_t = X_{ht}$. Then, Y_t solves

$$
dY_t = hb(Y_t) + \sqrt{2}hdB_t \quad Y_0 = x_0.
$$

Now, define the first hitting times, by

$$
\tau_Y := \inf\{t \geq 0 : Y_t \in \partial\Omega\} \quad \tau_X := \inf\{t \geq 0 : X_t \in \partial\Omega\} = h\tau_Y.
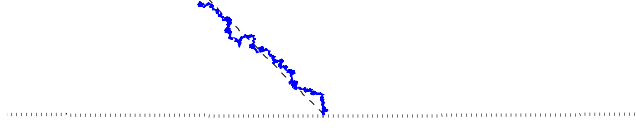\tag{9.2}
$$

We prove the following proposition,

FIGURE 5. The figure shows a sample of the diffusion process X_t with $b(X_t) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $h = 10^{-4}$, and initial condition $X_0 = (0, h)$. The dotted line shows the boundary of a disk tangent to $y = 0$ of radius $1/2$ and the dashed line shows the path of the ode with no noise. The boundary is shown for $y < 10h$.

Proposition 4. *Let X_t and τ_X be defined as in (9.1) and (9.2) respectively. Then, for each $x_0 \in \partial\Omega_+$ (where $X = -b$ in (1.3)), and $\gamma > 0$, there is an h dependent family of $x(h) = O(h^\gamma)$ such that for h small enough,*

$$h \log E_{x_0 + x(h)} e^{\tau_X / h} \geq C h \log h.$$

Moreover, if $\partial\Omega$ and b are real analytic near x_0, there is a $\delta > 0$ such that

$$h \log E_{x_0 + x(h)} e^{\tau_X / h} \geq \delta$$

and such that for every $a > 1$, there is a sequence $s(h) > \delta/2$ with

$$c_a \min(e^{-a(s(h) - \delta)/h}, 1) \leq P\left(\tau_X \geq \frac{\delta}{2\lambda}\right).$$

Proof. It is a standard result of probability theory [7, Section 1.5] that the operator

$$L := -(hD)^2 + i\langle b, hD\rangle$$

is associated to this diffusion process Y_t. Let $\lambda_1(L)$ be the principal eigenvalue of L. We have that $0 < c \leq \lambda_1(L)$ where c is independent of h and, for $\lambda < \lambda_1(L)$, we have that the solution, u to

$$(9.3) \qquad \begin{cases} (-L - \lambda)u = ((hD)^2 + i\langle -b, hD\rangle - \lambda)u = f & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

has

$$(9.4) \qquad u(x) = \mathbb{E}_x \int_0^{\tau_Y} f(Y_t) e^{\lambda t} dt$$

where $\mathbb{E}_x$ is the expectation given that $Y_0 = x$.

Now, by Proposition 2, there are quasimodes for (9.3) if $0 < \lambda$. The quasimodes are concentrated near x_0 for x_0 in the subset of the boundary illuminated by $-b$. Let $\langle \mu, \nu(x_0)\rangle < 0$, $\epsilon(h) \leq Ch$. We change coordinates so that $\nu(x_0) = \partial x_1$ and observe that near the point x_0, these

quasimodes have

$$|u(\mu\epsilon(h) + x_0)| = \left| e^{-c|\mu'|^2\epsilon(h)^2/h} \right| \left| a(x\epsilon(h) + x_0)e^{ic_1\mu_1\epsilon(h)/h + O(\mu_1\mu'\epsilon(h)^2/h)} \right.$$

$$\left. - b(\mu\epsilon(h) + x_0)e^{ic_2\mu_1\epsilon(h)^{-1} + O(\mu_1\mu'\epsilon(h)^2/h)} \right|$$

$$= (1 + O(\epsilon(h))) \left| (e^{ic_1\mu_1\epsilon(h)/h} - e^{ic_2\mu_1\epsilon(h)/h}) \right| \geq C_\mu\epsilon(h)/h$$

Therefore, for every μ with $\langle \mu, \nu(x_0) \rangle < 0$ and $\epsilon(h) \leq Ch$, $|u(\mu\epsilon(h) + x_0)| \geq C\epsilon(h)/h$. Now, applying this in (9.4), we have

$$C\frac{\epsilon(h)}{h} \leq \left| \mathbb{E}_{x_0+\mu\epsilon(h)} \int_0^{\tau_Y} f(Y_t)e^{\lambda t}dt \right| = \left| \mathbb{E}_{x_0+\mu\epsilon(h)} \int_0^{\tau_Y} (-L - \lambda)u(Y_t)e^{\lambda t}dt \right|$$

$$\leq \|(-L - \lambda)u\|_{L^\infty} \frac{1}{|\lambda|} \mathbb{E}_{x_0+\mu\epsilon(h)} \left(e^{\lambda\tau_Y} - 1 \right).$$

If $\partial\Omega$ and b are real analytic near x_0, we have $\|(-L - \lambda)u\|_{L^\infty} = O(e^{-\delta/h})$ which yields

$$\frac{\epsilon(h)}{h}e^{\delta/h} \leq \mathbb{E}_{x_0+\mu\epsilon(h)}e^{\lambda\tau_Y} = \mathbb{E}_{x_0+\mu\epsilon(h)}e^{\lambda\tau_X/h}$$

and if $\partial\Omega$ or b is only C^∞ near x_0, $\|(-L - \lambda)u\|_{L^\infty} = O(h^\infty)$ and hence, for all $N > 0$ there exists c_N such that

$$c_N\epsilon(h)h^{-N} \leq \mathbb{E}_{x_0+\mu\epsilon(h)}e^{\lambda\tau_Y} = \mathbb{E}_{x_0+\mu\epsilon(h)}e^{\lambda\tau_X/h}.$$

Thus, if there exists $\gamma > 0$ such that $\epsilon(h) > Ch^\gamma$, we have, possibly with a different δ,

$$(9.5) \qquad E_{x_0+\mu\epsilon(h)}e^{\lambda\tau_X/h} \geq \begin{cases} e^{\delta/h} & \partial\Omega, b \text{ analytic near } x_0, \\ c_N h^{-N} & \partial\Omega, b \ C^\infty \text{ near } x_0. \end{cases}$$

This gives the first two statements in Propostion 4.

Remark: Notice also, applying the standard small noise perturbation results that can be found, for example, in [7, Theorem 2.3] to a domain $\Omega_\delta \supset \Omega$ with $B(x_0, \delta) \subset \Omega_\delta$, and defining τ_X^δ the corresponding hitting time, that we have for some $C > 0$

$$\mathbb{E}_{x_0+\mu\epsilon(h)}e^{\lambda\tau_X/h} \leq \mathbb{E}_{x_0+\mu\epsilon(h)}e^{\lambda\tau_X^\delta/h} \leq e^{C/h}.$$

We now prove the second part of the proposition. Compute, using that fact that $\tau_X \geq 0$ and making the change of variables $s = h \log x$,

$$E_{x_0+\mu\epsilon(h)}e^{\lambda\tau_X/h} = \int_0^\infty P\left(e^{\lambda\tau_X/h} \geq x \right) dx = \frac{1}{h} \int_0^\infty e^{s/h} P\left(\tau_X \geq s\lambda^{-1} \right) ds.$$

Hence, in the analytic case,

$$\frac{1}{h} \int_0^\infty e^{s/h} P\left(\tau_X \geq s\lambda^{-1} \right) ds \geq e^{\delta/h}$$

and we have that

$$\frac{1}{h} \int_0^\infty e^{(s-\delta)/h} P\left(\tau_X \geq s\lambda^{-1} \right) ds \geq 1.$$

Now, making the change of variables $t = (s - \delta)/h$.

$$\int_{-\delta/h}^\infty e^t P\left(\tau_X \geq \lambda^{-1}(ht + \delta) \right) dt = O(e^{-\delta/(2h)}) + \int_{-\delta/(2h)}^\infty e^t P\left(\tau_X \geq \lambda^{-1}(ht + \delta) \right) dt.$$

Thus, choosing $g(t) \in L^1(e^t dt)$ with $g(t) > 0$, we have for h small enough, there is a $c > 0$ such that

$$c \leq \int_{-\delta/(2h)}^{\infty} e^t P\left(\tau_X \geq \lambda^{-1}(ht+\delta)\right) dt \leq \left\| P\left(\tau_X \geq \lambda^{-1}(ht+\delta)\right) g^{-1}(t) \right\|_{L^\infty} \int_{-\delta/2h}^{\infty}$$

and hence

$$\frac{c}{\|g\|_{L^1(e^t dt)}} \leq \left\| P\left(\tau_X \geq \lambda^{-1}(ht+\delta)\right) g^{-1}(t) \right\|_{L^\infty}.$$

That is, letting $ht + \delta = s$, there exists $s(h) \geq \delta/2$ such that

$$\frac{c}{\|g\|_{L^1(e^t dt)}} g\left(\frac{s(h)-\delta}{h}\right) \leq P\left(\tau_X \geq s(h)\lambda^{-1}\right) \leq P\left(\tau_X \geq \frac{\delta}{2\lambda}\right).$$

Fixing $a > 1$ and letting $g(t) = \min(e^{-at}, 1)$ gives the last part of Proposition 4. $\qquad\square$

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